where

$$D = \begin{bmatrix} \int M_{00}(x)dx & 0 \\ 0 & \operatorname{diag}\left\{ \int \begin{bmatrix} M_{00} & M_{0j} \\ M_{j0} & M_{jj} \end{bmatrix} dx \right\}_{j=1,\ldots,d} \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 0^{\top} \\ 0 & \Pi \end{bmatrix}.$$

Since $g'(\tilde{\beta}_{K-1}(\cdot))^2 V(\tilde{\beta}_{K-1}(\cdot))$ is bounded and $\tilde{\beta}_{K-1}(\cdot)$ is bounded away from 0, one can obtain that D^{-1} is bounded. Using the same technique, we can prove that Π is one-to-one and onto, and $\|\Pi\| \leq d\|k\|$. Hence Π has a bounded inverse, which implies that A also has, i.e. there exists a constant c_1 such that $\|\tilde{F}_K^{(1)}(\tilde{\beta}_K^{b0c})\| < c_1$.

The other conditions are verified as follows. To prove that $\tilde{F}_K^{(1)}$ satisfies the Lipschitz condition $\|\tilde{F}_K^{(1)} - \tilde{F}_K^{(1)\prime}\| \leq c_2\|\cdot - \cdot'\|$, by the decomposition (2), it is sufficient to prove the boundedness of D and A, which is guaranteed by the smoothness and boundedness of $g'(\tilde{\beta}_{K-1}(\cdot))^2 V(\tilde{\beta}_{K-1}(\cdot))$ and $\tilde{\beta}_{K-1}(\cdot)$. Finally, by the uniform continuity of $\tilde{F}_K$ and the fact $\tilde{F}_K - \tilde{\beta}_K = 0$, there exists a positive constant r such that $\sup_{\beta \in B_r(\tilde{\beta}_K)} \|\tilde{F}_K - \cdot\| < (2c_1^2c_2)^{-1}$. This proves that if $\tilde{\beta}_K^{b0c} \in B_r(\tilde{\beta}_K)$, then $\|f - \tilde{F}_K^{(1)}(\tilde{\beta}_K^{b0c})g^{-1} - \tilde{F}_K(\tilde{\beta}_K^{b0c})\| < (2c_1c_2)^{-1}$. Note that M is close to $\overline{Q}_K$, the two norms $\|\cdot\|_M$ and $\|\cdot\|_{\overline{Q}_K}$ are equivalent. Conclusion (1) now follows from Lemma 1. Theorem 1 (2) follows from the property of Hilbert-Schmidt operators Π_j; see Theorem 4.B in Appendix 4 of Bickel et al. (1993) and Theorem 4 of Yu et al. (2008) for details. □

S.3.2 Proof of Lemma 2-3

We first prove Lemma 2. Recall that $\cdot$ is defined in (23). The proof is based on the decomposition of $\cdot$.

Proof. We further decompose $_j(x_j) = \ ^A_j(x_j) + \ ^B_j(x_j)$; $_{1j}(x_j) = \ ^A_{1j}(x_j) + \ ^B_{1j}(x_j)$ that

$$\begin{bmatrix} \ ^A_j(x_j) \\ \ ^A_{1j}(x_j) \end{bmatrix} = \begin{bmatrix} \ _j(x_j) \\ \ _{1j}(x_j) \end{bmatrix} - \begin{bmatrix} E(\ _j(x_j) \mid X_1;::::;X_K) \\ E(\ _{1j}(x_j) \mid X_1;::::;X_K) \end{bmatrix};$$

$$\begin{bmatrix} \ ^B_j(x_j) \\ \ ^B_{1j}(x_j) \end{bmatrix} = \begin{bmatrix} E(\ _j(x_j) \mid X_1;::::;X_K) \\ E(\ _{1j}(x_j) \mid X_1;::::;X_K) \end{bmatrix}:$$

We omit the constants in the functions and de ne the spaces

$$\overline{H}^0(Q) = \ f = (f_1(x_1);::::;f_d(x_d);f_{11}(x_1);f_{1d}(x_d))^> : \int (Q_{00}f_j + Q_{0j}f_{1j})dx = 0 \text{ for } j = 1;::::;d \ ;$$

$$\overline{H}^0_j(Q) = \{f \ 2 \ \overline{H}^0(Q) : f \text{ depends only on } x_j\}:$$

In this proof, we consider $2d$-dimensional functions. Let $_j$ be the projection operator from $\overline{H}^0(Q_K)$ onto $\overline{H}^0_j(Q_K)$. Then for any $\ 2 \ \overline{H}^0(Q_K)$,

$$_j \ _\ell(x_\ell) = f_\ell(x_\ell); \qquad _{1j} \ _{1\ell}(x_\ell) = f_{1\ell}(x_\ell);$$

where for $` \neq j$, $f_\ell(x_\ell) = \ _\ell(x_\ell)$ and $f_{1\ell}(x_\ell) = \ _{1\ell}(x_\ell)$, and

$$f_j(x_j) = g_j(x_j) - \int g_j(x_j)\overline{Q}_{K,00}(x)dx;$$

$$\begin{bmatrix} g_j(x_j) \\ f_{1j}(x_j) \end{bmatrix} = \sum_{\ell \neq j} \left(\int \begin{bmatrix} \overline{Q}_{K,00} & \overline{Q}_{K,0j} \\ \overline{Q}_{K,j0} & \overline{Q}_{K,jj} \end{bmatrix} dx \ _j \right) \int \begin{bmatrix} \overline{Q}_{K,00} & \overline{Q}_{K,0\ell} \\ \overline{Q}_{K,j0} & \overline{Q}_{K,j\ell} \end{bmatrix} \begin{bmatrix} \ _\ell(x_\ell) \\ \ _{1\ell}(x_\ell) \end{bmatrix} dx \ _j:$$

Then de ne $T = \ _1 \quad _d$. Recall that $!_{ki}(x; \ )$ plays the same role of $K_h(X_{ki} \ x)$ in Mammen (1999). With assumption (A2) on the smoothness of link function g, conclusions of AM can be carried over to GAM and we can obtain that T is smaller than $\ < 1$ w.p.1. Let $_j(x_j) = \int \sum_{k=1}^K P_{k,0}(\ ; e_{kjK})dx \ _j$, $_{1j}(x_j) = \int \sum_{k=1}^K P_{k,j}(\ ; e_{kjK})dx \ _j$, $_{0,j}(x_j) = \int_0 \overline{Q}_{K,00}(x)dx \ _j$ and $_{0,1j} = \int_0 \overline{Q}_{K,0j}(x)dx \ _j$. Then we have the following expression for $\ , \ = T \ + \ $, where $\ = \ + \ _0$. By iterative applications, we have

$^{s} = \sum_{i=0}^{1} T^{i} \ ^{s}$ for $s = A; B$. Based on these stochastic expansions, one can use the techniques in the proof of Theorem 1{2 in Mammen and Nielsen (1999) to obtain the following result: for x_j 2 (0; 1),

$$\sup \ {}_{j}^{A}(x_j) \qquad {}_{j}(x_j) \qquad {}_{0} \quad = o_p(N_K^{2/5});$$

$$\sup \ {}_{j}^{B}(x_j) \quad b_j(x_j) \ = o_p(N_K^{2/5});$$

and the asymptotic distribution of follows from the standard theory of kernel smoothing.

□

Now we present the proof of Lemma 3, which is a product of the pre-speci ed bandwidth order in Assumption (A5).

Proof. Noting that F_K di ers from $\hat{F}_K$ only in the bandwidths, the proof of Lemma 6 of Yu et al. (2008) can be carried over by substituting $_{Kj,i}$ for h_j^i. By Assumption (A5), we have

$$kF_K \ _K k_{p,0} = o_p \ N_K^{2/5} \ ; \quad kF_K \ _K k_1 = o_p \ N_K^{2/5} \ :$$

We note that $\tilde{F}_K$ is the linear approximation of F_K, then $kF_K \ _K k \ ' \ k\tilde{F}_K \ _K k$ for norms $k \ k_{p,0}$ and $k \ k_1$, which gives

$$k\tilde{F}_K \ _K k_{p,0} = o_p \ N_K^{2/5} \ ; \quad k\tilde{F}_K \ _K k_1 = o_p \ N_K^{2/5} \ : \tag{3}$$

Finally, we can use the same techniques in proof of Theorem 1 by substituting $_K$ for e_K^{bmc} to prove that the su cient conditions of Lemma 1 hold when setting $_0 = \ _K$. This implies that $_K$ must be close to e_K at the same rate as (3), which complete the proof. □

S.3.3 Proof of Proposition 1

Proof. Using the same technique in the proof of Theorem 2, we can derive that

$$E f(e_{Kj} \qquad {}_{j})^2 g = \ A_{1j} \ {}_{Kj,2}^{\theta} + A_{2j} N_K^{1} \ {}_{Kj,\ 5}^{\theta \ 2} + A_{3j} N_K^{2} \ {}_{Kj,\ 9}^{\theta}$$

$$+ o_p \quad {}^{\theta}_{Kj,2} + N_K^{1} \, {}^{\theta}_{Kj,\ 5}{}^{2} + N_K^{2} \, {}^{\theta}_{Kj,\ 9} \ ;$$

where $A_{1j}; A_{2j}; A_{3j}$ are constants and

$${}^{\theta}_{Kj,i} = \frac{1}{N_K} \sum_{k=1}^{K} n_k (\hat{e}^{\theta}_{kjK,j})^i; \ i = 2; \ 5; \ 9:$$

Using the same argument as S.3.4, when (21) and (22) hold, the online estimates $\hat{e}_{Kj}$ satis es $\hat{e}_{Kj} \quad {}_j = O_p \ N_K^{2/7}$. Substituting the expression of $\hat{\theta}_{Kj}$ in (16) and h_{Kj} in (14), we obtain

$$\frac{\hat{\theta}_{Kj} \quad h_{Kj}}{h_{Kj}} = \frac{1}{5} \quad \frac{{}_j}{\hat{e}_{Kj}} \left(\frac{\hat{e}^2_{K,j} \quad {}^2_j}{{}^2_j} \quad \frac{1}{5} \frac{\hat{e}_{Kj} \quad {}_j}{{}_j} \right)^{\frac{1}{5}}.$$

When $h^{\sigma}_{Kj} = O_p(N_K{}^{1/5})$, $\hat{e}^2_{Kj}(u) \quad {}^2_j(u) = O_p(N_K{}^{2/5})$ and then $\hat{e}^2_{K,j} \quad {}^2_j = O_p(N_K{}^{2/5})$, and hence the convergence rate of $\hat{\theta}_{Kj}$ is dominated by $\hat{e}_{Kj} \quad {}_j$. □

S.3.4 Proof of Theorem 3

Proof. This result follows directly from Theorem 4 in Yang and Yao (2022) based on the oracle properties of the proposed estimate in Theorem 2. The optimal bandwidth h_{Kj} as in (14) is strictly decreasing with K. Thus the candidate sequence shall be decreasing with ${}_{kj1} = \hat{\theta}_{kj}$. There is no e ciency loss if $\hat{e}_{kjK,j} = \hat{h}_{Kj}$ which implies that the optimal ${}_{lj}$ shall make $\hat{e}_{kjK,j}$ as close to $\hat{h}_{Kj}$ as possible. With the convergence of $\hat{\theta}_{kj}$, we have

$$\frac{\hat{e}_{kjK,j}}{\hat{h}_{Kj}} = g(l) \quad \frac{N_k}{N_K} \quad {}^{\frac{1}{5}} + O_p \quad N_k \quad {}^{\frac{2}{7}} \ :$$

Write $g(l) = fg(l)^{1/\lambda} g^{\lambda}$ and note that N_k/N_K grows linearly, then shall be 1=5 and the optimal $g(l)^{1/\lambda}$ shall be linear between $(0; 1)$. Hence, the optimal is

$${}_{Kj\ell} = \quad \frac{L \quad \ell + 1}{L} \quad {}^{\frac{1}{5}} \hat{\theta}_{Kj}:$$

Next we derive the asymptotic lower bound for the relative e ciency. Writing $\hat{h}_{Kj}$ = $h_{Kj}\{1 + O_p(N_K^{-2/7})\}$ and using the expressions of h_{Kj} as in (14), one can derive

$eff(e_{Kj})$

$$+\frac{4}{5}\left(\frac{5}{6}\right)^{\frac{1}{5}}\left(\frac{1+\frac{1}{5}}{0}\right)^{\frac{1}{5}}+1\qquad_0\;+O_p\left(\frac{N_K^{\frac{1}{5}}}{L}+N_K^{\frac{2}{7}}\right)^{!}: \tag{5}$$

The property of breakpoint K_0 also guarantee that

$$\frac{1}{L}\sum_{k=1}^{\frac{1}{5}K_0}\frac{n_k}{N_{K_0}}\theta_{kj}\qquad\theta_{Kj};$$

holds with probability 1 when K tends large, which is equivalent to

$$\frac{1}{L}\sum_{k=1}^{\frac{1}{5}K_0}\left(\frac{n_k}{N_{K_0}}\left(\frac{c_kN_k^{\frac{1}{5}}}{c_KN_K^{\frac{1}{5}}}\right)\right)\qquad 1:$$

Under Assumption (A6), we obtain

$$\frac{N_{K_0}}{N_K}\qquad\left(\frac{5}{4}\right)^5\frac{1}{L}+O_p(N_K^{\frac{2}{7}});$$

i.e. $_0\ 2\ [0;(5=4)^5=L]$ holds with probability 1 when K tends large. Note that (5) is strictly increasing with respect to $_0$ on this domain. Hence we have

$$eff(\theta_{Kj})^{\ 1}\quad 1+c_1L^{\ 1}+c_2L^{\ 2};$$

where

$$c_1=5^3=(3\quad 2^5)+5^6=(6\quad 4^5)\quad 6=5\quad(5=4)^5\quad 0{:}183$$

and

$$c_2=f5^7=(6\quad 4^6)\quad(5=4)^5g^2=5\quad 0{:}0032{:}$$

This completes the proof of Theorem 3. □

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