where

$$D = \begin{bmatrix} \int M_{00}(x)dx & 0^{\top} \\ 0 & \mathrm{diag}\left(\left\{\int \begin{bmatrix} M_{00} & M_{0j} \\ M_{j0} & M_{jj} \end{bmatrix} dx\right\}_{j=1,\ldots,d}\right) \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 0^{\top} \\ 0 & \Pi \end{bmatrix}.$$

Since $g^{0}(\mathbf{e}_{K\ 1}(\ ))^{2}V(\mathbf{e}_{K\ 1}(\ ))$ is bounded and $\mathbf{e}_{K\ 1}(\ )$ is bounded away from 0, one can obtain that $D^{\ 1}$ is bounded. Using the same technique, we can prove that Π is one-to-one and onto, and $k\Pi\ k \quad dk\ k$. Hence Π has a bounded inverse, which implies that A also has, i.e. there exists a constant c_1 such that $k\mathbf{F}_K^{(1)}(\mathbf{e}_K^{b0c})k < c_1$.

The other conditions are veri ed as follows. To prove that $\mathbf{F}_K^{(1)}$ satis es the Lipschitz condition $k\mathbf{F}_K^{(1)} \quad \mathbf{F}_K^{(1)}\ ^{0}k \quad c_2k \quad ^{0}k$, by the decomposition (2), it is su cient to prove the boundedness of D and A, which is guaranteed by the smoothness and boundedness of $g^{0}(\mathbf{e}_{K\ 1}(\ ))^{2}V(\mathbf{e}_{K\ 1}(\ ))$ and $\mathbf{e}_{K\ 1}(\ )$. Finally, by the uniform continuity of $\mathbf{F}_K$ and the fact $\mathbf{F}_K\,\mathbf{e}_K = 0$, there exists a positive constant r such that $\sup_{\beta 2 B_r(\widetilde{\beta}_K)} k\mathbf{F}_K\ \ k < (2c_1^2c_2)^{\ 1}$. This proves that if $\mathbf{e}_K^{b0c}\ 2\ B_r(\mathbf{e}_K)$, then $kf\mathbf{F}_K^{(1)}(\mathbf{e}_K^{b0c})g^{\ 1}\mathbf{F}_K(\mathbf{e}_K^{b0c})k < (2c_1c_2)^{\ 1}$. Note that M is close to $\overline{Q}_K$, the two norms $k\ \ k_M$ and $k\ \ k_{\overline{Q}_K}$ are equivalent. Conclusion (1) now follows from Lemma 1. Theorem 1 (2) follows from the property of Hilbert-Schmidt operators Π_j; see Theorem 4.B in Appendix 4 of Bickel et al. (1993) and Theorem 4 of Yu et al. (2008) for details. □

S.3.2 Proof of Lemma 2-3

We rst prove Lemma 2. Recall that is de ned in (23). The proof is based on the decomposition of .

Proof. We further decompose $_j(x_j) = {}^A_j(x_j) + {}^B_j(x_j)$; $_{1j}(x_j) = {}^A_{1j}(x_j) + {}^B_{1j}(x_j)$ that

$$\begin{bmatrix} {}^A_j(x_j) \\ {}^A_{1j}(x_j) \end{bmatrix} = \begin{bmatrix} {}_j(x_j) \\ {}_{1j}(x_j) \end{bmatrix} \begin{bmatrix} E({}_j(x_j) \mid X_1; \ldots; X_K) \\ E({}_{1j}(x_j) \mid X_1; \ldots; X_K) \end{bmatrix};$$

$$\begin{bmatrix} {}^B_j(x_j) \\ {}^B_{1j}(x_j) \end{bmatrix} = \begin{bmatrix} E({}_j(x_j) \mid X_1; \ldots; X_K) \\ E({}_{1j}(x_j) \mid X_1; \ldots; X_K) \end{bmatrix}.$$

We omit the constants in the functions and de ne the spaces

$$\overline{H}^0(Q) = \quad f = (f_1(x_1); \ldots; f_d(x_d); f_{11}(x_1); f_{1d}(x_d))^{>} : \int (Q_{00} f_j + Q_{0j} f_{1j}) dx = 0 \text{ for } j = 1; \ldots; d \quad ;$$

$$\overline{H}^0_j(Q) = f f \ 2 \ \overline{H}^0(Q) : f \text{ depends only on } x_j g:$$

In this proof, we consider $2d$-dimensional functions. Let $_j$ be the projection operator from $\overline{H}^0(Q_K)$ onto $\overline{H}^0_j(Q_K)$. Then for any $\ 2 \ \overline{H}^0(Q_K)$,

$$_j \ _\ell(x_\ell) = f_\ell(x_\ell); \qquad _{1j} \ _{1\ell}(x_\ell) = f_{1\ell}(x_\ell);$$

where for $\ell \neq j$, $f_\ell(x_\ell) = {}_\ell(x_\ell)$ and $f_{1\ell}(x_\ell) = {}_{1\ell}(x_\ell)$, and

$$f_j(x_j) = g_j(x_j) \int g_j(x_j) \overline{Q}_{K,00}(x) dx;$$

$$\begin{bmatrix} g_j(x_j) \\ f_{1j}(x_j) \end{bmatrix} = \sum_{\ell \neq j} \left(\int \begin{bmatrix} \overline{Q}_{K,00} & \overline{Q}_{K,0j} \\ \overline{Q}_{K,j0} & \overline{Q}_{K,jj} \end{bmatrix} dx \ {}_j \right)^{-1} \int \begin{bmatrix} \overline{Q}_{K,00} & \overline{Q}_{K,0\ell} \\ \overline{Q}_{K,j0} & \overline{Q}_{K,j\ell} \end{bmatrix} \begin{bmatrix} {}_\ell(x_\ell) \\ {}_{1\ell}(x_\ell) \end{bmatrix} dx \ {}_j.$$

Then de ne $T = \ _1 \quad _d$. Recall that $!_{ki}(x; \ )$ plays the same role of $K_h(X_{ki} \quad x)$ in Mammen (1999). With assumption (A2) on the smoothness of link function g, conclusions of AM can be carried over to GAM and we can obtain that T is smaller than < 1 w.p.1. Let $_j(x_j) = \int \sum_{k=1}^K P_{k,0}(\ ; e_{kjK}) dx \ {}_j$, $_{1j}(x_j) = \int \sum_{k=1}^K P_{k,j}(\ ; e_{kjK}) dx \ {}_j$, $_{0,j}(x_j) = \int_0 \overline{Q}_{K,00}(x) dx \ {}_j$ and $_{0,1j} = \int_0 \overline{Q}_{K,0j}(x) dx \ {}_j$. Then we have the following expression for $\ , \ = T \ + \ , \ \text{where} \ = \ + \ _0$. By iterative applications, we have

$_s = \sum_{i=0}^{1} T^i \ _s$ for $s = A; B$. Based on these stochastic expansions, one can use the techniques in the proof of Theorem 1{2 in Mammen and Nielsen (1999) to obtain the following result: for $x_j \ 2 \ (0; 1)$,

$$\sup \ _j^A(x_j) \quad _j(x_j) \quad _0 \ = o_p(N_K^{2/5});$$

$$\sup \ _j^B(x_j) \quad b_j(x_j) \ = o_p(N_K^{2/5});$$

and the asymptotic distribution of follows from the standard theory of kernel smoothing.

□

Now we present the proof of Lemma 3, which is a product of the pre-speci ed bandwidth order in Assumption (A5).

Proof. Noting that F_K di ers from $\hat{F}_K$ only in the bandwidths, the proof of Lemma 6 of Yu et al. (2008) can be carried over by substituting $_{Kj,i}$ for h_j^i. By Assumption (A5), we have

$$kF_K \ _K k_{p,0} = o_p \ N_K^{2/5} \ ; \quad kF_K \ _K k_1 = o_p \ N_K^{2/5} \ :$$

We note that $\tilde{F}_K$ is the linear approximation of F_K, then $kF_K \ _K k \ ' \ k\tilde{F}_K \ _K k$ for norms $k \ k_{p,0}$ and $k \ k_1$, which gives

$$k\tilde{F}_K \ _K k_{p,0} = o_p \ N_K^{2/5} \ ; \quad k\tilde{F}_K \ _K k_1 = o_p \ N_K^{2/5} \ : \tag{3}$$

Finally, we can use the same techniques in proof of Theorem 1 by substituting $_K$ for $\tilde{e}_K^{bmc}$ to prove that the su cient conditions of Lemma 1 hold when setting $_0 = \ _K$. This implies that $_K$ must be close to $\tilde{e}_K$ at the same rate as (3), which complete the proof. □

S.3.3 Proof of Proposition 1

Proof. Using the same technique in the proof of Theorem 2, we can derive that

$$E f(\tilde{e}_{Kj} \quad _j)^2 g = \ A_{1j} \ \theta_{Kj,2} + A_{2j} N_K^1 \ \theta_{Kj,\ 5}^{\ 2} + A_{3j} N_K^2 \ \theta_{Kj,\ 9}$$

$$+ o_p \quad {}^{\theta}_{Kj,2} + N_K^{1} \; {}^{\theta}_{Kj,\ 5}{}^{2} + N_K^{2} \; {}^{\theta}_{Kj,\ 9} \quad ;$$

where $A_{1j}; A_{2j}; A_{3j}$ are constants and

$$ {}^{\theta}_{Kj,i} = \frac{1}{N_K} \sum_{k=1}^{K} n_k (\mathrm{e}^{\theta}_{kjK,j})^i; \; i = 2; \quad 5; \quad 9:$$

Using the same argument as S.3.4, when (21) and (22) hold, the online estimates e_{Kj} satis es $\mathrm{e}_{Kj} \quad {}_j = O_p \; N_K^{2/7}$. Substituting the expression of $\hat{h}_{Kj}$ in (16) and h_{Kj} in (14), we obtain

$$\frac{\hat{h}_{Kj} \quad h_{Kj}}{h_{Kj}} = \frac{1}{5} \quad \frac{{}_j}{\mathrm{e}_{Kj}} \quad \left(\frac{\mathrm{e}^2_{K,j} \quad {}^2_j}{{}^2_j} \right)^{\frac{1}{5}} \quad \frac{1}{5} \frac{\mathrm{e}_{Kj} \quad {}_j}{{}_j}:$$

When $h^{\sigma}_{Kj} = O_p(N_K^{1/5})$, $\mathrm{e}^2_{Kj}(u) \quad {}^2_j(u) = O_p(N_K^{2/5})$ and then $\mathrm{e}^2_{K,j} \quad {}^2_j = O_p(N_K^{2/5})$, and hence the convergence rate of $\hat{h}_{Kj}$ is dominated by $\mathrm{e}_{Kj} \quad {}_j$. □

S.3.4 Proof of Theorem 3

Proof. This result follows directly from Theorem 4 in Yang and Yao (2022) based on the oracle properties of the proposed estimate in Theorem 2. The optimal bandwidth h_{Kj} as in (14) is strictly decreasing with K. Thus the candidate sequence shall be decreasing with ${}_{kj1} = \hat{h}_{kj}$. There is no e ciency loss if $\mathrm{e}_{kjK,j} = \hat{h}_{Kj}$ which implies that the optimal ${}_{lj}$ shall make $\mathrm{e}_{kjK,j}$ as close to $\hat{h}_{Kj}$ as possible. With the convergence of $\hat{h}_{kj}$, we have

$$\frac{\mathrm{e}_{kjK,j}}{\hat{h}_{Kj}} = g(l) \quad \left(\frac{N_k}{N_K} \right)^{\frac{1}{5}} + O_p \quad N_k^{\frac{2}{7}} \quad :$$

Write $g(l) = fg(l)^{1/\lambda} g^{\lambda}$ and note that N_k/N_K grows linearly, then shall be 1/5 and the optimal $g(l)^{1/\lambda}$ shall be linear between $(0; 1)$. Hence, the optimal is

$${}_{Kj\ell} = \left(\frac{L \quad \ell + 1}{L} \right)^{\frac{1}{5}} \hat{h}_{Kj}:$$

Next we derive the asymptotic lower bound for the relative e ciency. Writing $\hat{h}_{Kj}$ = $h_{Kj}\{1 + O_p(N_K^{2/7})\}$ and using the expressions of h_{Kj} as in (14), one can derive

$$eff(\hat{e}_{Kj})$$

$$+ \frac{4}{5} \quad \frac{5}{6} \quad L^{\frac{1}{5}} \quad {}_0^{1+\frac{1}{5}} + 1 \qquad {}_0 \quad + O_p \quad \frac{N_K{}^{\frac{1}{5}}}{L} + N_K{}^{\frac{2}{7}} \quad ! \quad : \tag{5}$$

The property of breakpoint K_0 also guarantee that

$$\frac{1}{L} \quad {}^{\frac{1}{5}} \sum_{k=1}^{K_0} \frac{n_k}{N_{K_0}} \theta_{kj} \quad \theta_{Kj};$$

holds with probability 1 when K tends large, which is equivalent to

$$\frac{1}{L} \quad {}^{\frac{1}{5}} \sum_{k=1}^{K_0} \frac{n_k}{N_{K_0}} \left(\frac{c_k N_k{}^{\frac{1}{5}}}{c_K N_K{}^{\frac{1}{5}}} \right) \quad 1:$$

Under Assumption (A6), we obtain

$$\frac{N_{K_0}}{N_K} \quad \frac{5}{4} \quad {}^5 \frac{1}{L} + O_p(N_K{}^{\frac{2}{7}});$$

i.e. ${}_0 \; 2 \; [0; (5/4)^5/L]$ holds with probability 1 when K tends large. Note that (5) is strictly increasing with respect to ${}_0$ on this domain. Hence we have

$$eff(\theta_{Kj}) \quad {}^1 \quad 1 + c_1 L \quad {}^1 + c_2 L \quad {}^2;$$

where

$$c_1 = 5^3/(3 \quad 2^5) + 5^6/(6 \quad 4^5) \quad 6/5 \quad (5/4)^5 \quad 0.183$$

and

$$c_2 = \{5^7/(6 \quad 4^6) \quad (5/4)^5 g^2/5 \quad 0.0032.$$

This completes the proof of Theorem 3. □

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