# Supplementary Material to "Intrinsic Riemannian Functional Data Analysis for Sparse Longitudinal Observations"

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# S.1 Additional Background on Riemannian Manifold

<sup>5</sup> Here we provide formal definitions related to Riemannian manifolds, starting with perhaps the most basic geometric space — the topological space.

**Definition S.1** (Topological space). A topological space is a set and a class of subsets of the set, called the open sets, such that the class contains the empty set and is closed under the formation of arbitrary unions and finite intersections. Such a class is called a topology.

- The most commonly seen topological space is  $\mathbb{R}$  together with the standard (but often not explicitly mentioned) topology that contains all open intervals, as well as the *d*-dimensional Euclidean space  $\mathbb{R}^d$  together with the standard topology that contains all open balls. In real analysis, continuous functions play an important role. The concept of continuity can be generalized to functions defined on and/or taking values in general topological spaces.
- **Definition S.2** (Continuity and homeomorphism). A function  $f T_1 T_2$  between two topological spaces is continuous if to every open set B of  $T_2$ , the set  $f^{-1}(B) = \{x T_1 f(x) B\}$  is an open set of  $T_1$ . If f is continuous and bijective and its inverse is also continuous, then we say f is a homeomorphism between  $T_1$ and  $T_2$ . Two topological spaces are homeomorphic to each other if there exists a homeomorphism between them.
- When both  $T_1$  and  $T_2$  are  $\mathbb{R}$  with the standard topology, the continuity defined in the above coincides with the one via the - definition. For instance, it is well know that  $f(x) = x^3$  is a continuous function, and indeed, to every open interval (a, b),  $f^{-1}((a, b)) = (a^{1/3}, b^{1/3})$  is also an open interval, and this holds for all open sets of  $\mathbb{R}$ ; see Example 1 of Section 18 in Munkres (2000). A property of 9 Tf-316(holds)-317(for)]589.9626 20

connected if there are no holes passing through the space. For instance,  $\mathbb{R}^d$  and spheres are simply connected, while the torus is not. Here, connectedness, path-connectedness and simple connectedness are all topological properties, i.e., preserved under homeomorphisms.

There are topological spaces that locally resemble a Euclidean space  $\mathbb{R}^d$ , but globally may not be homeomorphic to  $\mathbb{R}^d$ . Such spaces are called topological manifolds. An example of topological manifolds is the surface of our earth that may be roughly parameterized by the two-dimensional sphere  $\mathbb{S}_r^2 = \{(x_1, x_2, x_3)$  $\mathbb{R}^3 \quad x_1^2 + x_2^2 + x_3^2 = r^2\}$  of radius r 6371km. Each small neighborhood of  $\mathbb{S}_r^2$  looks like a (subset of) twodimensional plane, and this is why our ancients perceived the flat earth model. However, there exists no

homeomorphism between  $\mathbb{S}_r^2$  and  $\mathbb{R}^d$  for any d.

**Definition S.3** (Topological manifold). A topological space  $\top$  is a topological manifold modeled on  $\mathbb{R}^d$ , if for each point  $p \ \top$  there exists an open set that contains p and is homeomorphic to  $\mathbb{R}^d$ . Here, d is called the dimension of  $\top$ .

In the above example of  $\mathbb{S}_{r}^2$ , let  $U_1 = \mathbb{S}_{r}^2 \{(r, 0, 0)\}$  and  $U_2 = \mathbb{S}_{r}^2 \{(-r, 0, 0)\}$ . Then every point in  $\mathbb{S}_{r}^2$  falls into one of these two open sets. Moreover, both  $U_1$  and  $U_2$  are homeomorphic to  $\mathbb{R}^d$ , with the following corresponding homeomorphisms

$${}_{1}(X_{1}, X_{2}, X_{3}) = \frac{\Gamma}{\Gamma - X_{1}}(X_{2}, X_{3}) \quad \mathbb{R}^{2}$$
  
 
$${}_{2}(X_{1}, X_{2}, X_{3}) = \frac{\Gamma}{\Gamma + X_{1}}(X_{2}, X_{3}) \quad \mathbb{R}^{2}.$$

<sup>45</sup> This formally shows that  $\mathbb{S}^2_r$  is a topological manifold of dimension 2.

Due to the local resemblance between  $\mathbb{R}^d$  and a topological manifold, one might parameterize a local neighborhood of the topological manifold by using  $\mathbb{R}^d$ , as we did in the above for neighborhoods  $U_1$  and  $U_2$  of  $\mathbb{S}_r^2$ . Intuitively, the map  $_1$  (resp.  $_2$ ) assigns each point in  $U_1$  (resp.  $U_2$ ) a coordinate in  $\mathbb{R}^2$ . Since  $U_1 \quad U_2 = \mathbb{S}_r^2$ , each point gets a coordinate. However, for those points in  $U_1 \quad U_2$ , such as the points in the equator, are assigned two coordinates, one from  $_1$  and the other from  $_2$ . In this case, one can obtain the coordinate under  $_1$  if we know its coordinate under  $_2$ , and vice versa. For example, for  $(x_1, x_2, x_3) \quad U_1 \quad U_2$ , if its coordinate under  $_1$  is  $(y_1, z_1) \quad \mathbb{R}^2$ , then its coordinate under  $_2$  is  $(y_2, z_2) = (2 \quad 1^{-1})(y_1, z_1)$ , where represents the composition of functions, i.e.,  $(f \quad g)(x) = f(g(x))$  for generic functions f, g and argument x. Moreover,  $_2 \quad 1^{-1}$ 



Figure S.1: Illustration of the chart and transition map.

The pair (U, ) or sometimes itself is called a **chart** (or **coordinate map**). One can check that compatibility defines an equivalence relation among atlases, and a maximal atlas is simply the union of atlases within the same equivalence class. Therefore, a di erentiable manifold is essentially completely determined by an atlas, in the sense that any compatible atlas gives to the same di erentiable manifold and incompatible atlases result in distinct di erentiable manifolds. In light of this, in practice, we can describe a di erentiable manifold simply by providing an atlas. For instance, in the example of  $\mathbb{S}^2_r$ , the collection  $\mathscr{A} = \{(U_1, 1), (U_2, 2)\}$  forms a  $C^{\infty}$ -atlas that turns  $\mathbb{S}^2_r$  into a smooth manifold.

- Let M be a *d*-dimensional di erentiable manifold in the sequel. One merit of the di erentiable manifold is that we can discuss regularity of functions taking values in a di erentiable manifold. For instance, for an interval /  $\mathbb{R}$ , the function / M, which is also called a curve on M, is di erentiable at *t* / if the function /  $\mathbb{R}^d$  is di erentiable at *t* for a chart (*U*, ), and thus all charts, such that (*t*) *U* in the maximal atlas associated with M. The derivative, denoted by '(*t*), measuring the velocity of the curve
- at *t*, has di erent representations in di erent charts. However, once we know its representation in one chart, we can then obtain its representation in another chart via the transition maps. In this sense, the velocity of at *t* is essentially well defined, and is denoted by '(*t*). Note that this sense of "well-definedness" applies generally to other concepts and quantities in di erential geometry, that is, a manifold-related concept, such as the di erentiability of a curve, is well defined if it holds for all relevant charts, and a manifold-related quantity, such the derivative of a di erentiable curve at *t*, is well defined if its coordinate in one chart can be determined from the coordinate under another chart.

Consider all di erentiable curves 1 M with (t) = p. They may give rise to di erent derivatives '(t). If we fix any chart (U, ) with p U, then each '(t) is represented by ( )'(t) which is a vector in  $\mathbb{R}^d$  and all possible values of ( )'(t) form exactly the vector space  $\mathbb{R}^d$ . Therefore, with  $\frac{t}{d}$ denoting the collection of di erentiable curves 1 M such that (t) = p, we can view the space  $\binom{t}{p}$  as a vector space with the vector addition  $\binom{t}{1}(t) + \binom{t}{2}(t)$  defined to be  $\binom{t}{3}(t) = T_p M$  $T_p M = \{ \prime(t) \}$  $_{2})'(t) = (3)'(t)$ , and the scalar multiplication a'(t) defined to be '(t)such that (  $_{1})'(t) + ($ )'(t), where , 1, 2, 3,  $f_p^t$  and  $a \in \mathbb{R}$ . Note that, the representation such that a( )'(t) = a(of both  $\frac{1}{1}(t) + \frac{1}{2}(t)$  and a'(t) in other charts is determined from its representation in  $(U, \cdot)$ , and thus they are well defined manifold-related quantities. The space  $T_p M$  is called the **tangent space** at p and its elements are called **tangent vectors** at *p*. Such (informal) definition also shows that  $T_p \mathbb{M}$  depends on *p* through its dependence on  $\frac{t}{p}$ . This agrees well with the physical meaning of the tangent vector '(t) that represents the direction and amount to move if one wants to get to (t + t) from (t) within an infinitesimal amount time *t*, i.e., '(t) encodes both the velocity and the base point *p*. In the chart (U, ), '(t) can be represented by (p), (

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domain A U, is a function mapping U into  $\mathbb{R}^{2d}$ . We say V is a smooth vector field if to each chart V is a smooth manifold map. An example of (smooth) vector fields is the vector field (U, ), the function 140 for the movement of air on Earth that represents the wind speed and direction at each location.

**Definition S.8** (Riemannian manifold). A Riemannian manifold is a smooth manifold M endowed with an inner product ,  $_p$  on  $T_pM$  for each p M such that, for any smooth vector fields  $V_1$ ,  $V_2$  on M, the function  $V_1(p)$ ,  $V_2(p)_p$  is a smooth manifold map defined on M. The inner products , p are collectively referred р to as Riemannian metric or Riemannian metric tensor.

For a Riemannian manifold, each of its tangent spaces is now an inner product space, and thus along a normed space with the induced norm  $v_p = v, v_p$  for  $v = T_p M$ . As a vector space, each tangent space is entitled to an independent basis. For Riemannian manifold, since each tangent space is an inner product space, it is also entitled to orthonormal basis. In this context, a frame refers to a map that assigns each point of the manifold an independent basis, and when all of such basis are orthonormal, we say the frame is

### an orthonormal frame.

The Riemannian metric also induces a (canonical) **distance**  $d_{\mathcal{M}}$  on the Riemannian manifold, as follows. M, the restriction to an interval [a, b] / is referred to as a segment of and is For a smooth curve / denoted by ([a, b]). The length of the segment ([a, b]) is defined by

$$(a,b) = \int_{a}^{b} (t) (t) dt.$$

Such definition can be extended to regular curves that are formed by connecting a finite number of smooth segments, i.e., the length of a piecewisely smooth segment is defined to be the sum of the lengths of its smooth segments. For a connected Riemannian manifold M, we can define the distance between two points p, q by

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 $d_{\mathcal{M}}(p,q) = \inf\{(a,b) \mid (a) = p, (b) = q, \text{ and } \text{ is a regular curve}\}.$ 

The distance  $d_{\mathcal{M}}$  then turns  $\mathbb{M}$  also into a metric space. If such metric space is complete, then we say M is a complete Riemannian manifold. By Hopf-Rinow theorem, on a connected complete Riemannian manifold, two points can be connected by a minimizing geodesic of which the length is exactly the distance of the two points. By definition, a geodesic is a curve I = M such that for all su ciently small > 0165 and all interior t I, ([t, t + ]) is a shortest path that connects (t) and (t + ), and (s, t) = as - tfor a constant a and all s, t 1. In words, geodesics are constant-speed curves that are locally shortest segments. Note that a geodesic may not be globally a shortest path connecting two points. For example,  $(t) = (0, \cos(2 t), \sin(2 t))$  for t = [0, 1] is a geodesic, but it is not a shortest path from () to (1) for [0, 1, 2). A minimizing geodesic refers to those geodesics / M such that ([s, t]) is a shortest any 170 path connecting (s) and (t) for all s, t = I.

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Remark S.1. In some textbooks, the term "curve" (and similarly, "geodesic") is sometimes used to denote I M on the manifold M, rather than the map . In this paper, we do not adopt this the image of practice. When we say a curve, we always refers to the map itself. One shall note that, the length of a curve is invariant to parameterization, i.e., if  $J \in M$  is another curve such that t = g(s) for a smooth function with g'(s) = (g(s)) for all s = g(J), then (g(a), g(b)) = (a, b) for all a, b = J. This is because,  $(g(a), g(b)) = \frac{g(b)}{g(a)}$  (t)  $(t) dt = \frac{b}{a}$  (g(s))  $(g(s)) dg(s) = \frac{b}{a}$  (s) (s) ds = (a, b).

Below we assume M is complete and connected. The Riemannian metric also induces the Riemannian exponential map for each point. For a unit tangent vector  $u = T_{\rho} N_{\rho}$ , let u(t) be the geodesic such that  $_{u}(0) = p$  and  $'_{u}(0) = u$ . As a starting point specified by p and an initial direction specified by the unit tangent vector u together uniquely determine a geodesic, the celebrated Hopf–Rinow theorem asserts that the following map  $\text{Exp}_{p}$  is well defined at each p M on the entire tangent space  $T_{p}M$ .

**Definition S.9** (Exponential map). The map  $\operatorname{Exp}_p(u) = \frac{u}{\|u\|_p} (u_p)$  for  $u \to T_p M$  is called the exponential map at p.

**Remark S.2.** Exponential maps might be defined also for incomplete Riemannian manifolds, but potentially only locally, i.e.,  $\text{Exp}_p u$  may be only defined for tangent vectors  $u = T_p N$  in a neighborhood of the zero tangent vector  $0 = T_p N$ .

For  $p \, M$  and  $u \, T_p M$ , the curve  $_u(t) = \exp_p(tu)$  is a geodesic with speed  $u_p$ . The **cut time**  $\mathfrak{c}(p, u)$  is defined to be  $t \, \mathbb{R}_+$  such that  $_u([0, t-])$  is a minimizing geodesic for any > 0 but  $_u([0, t])$ <sup>190</sup> is not, i.e.,  $\mathfrak{c}(p, u) = \sup\{t \, \mathbb{R}_+ \, _u([0, t]) \text{ is a minimizing geodesic with } _u(t) = \exp_p(tu)\}$ . Let  $\mathbb{E}_p = \{tu\}$ 

 $u = T_p M$ ,  $u_p = 1,0$   $t < \mathfrak{c}(p,u)$ }, which is a neighborhood of the zero tangent vector in  $T_p M$ , and define  $D_p = \{ Exp_p u = u = E_p \}$ . Then  $Exp_p$  is bijective between  $E_p$  and  $D_p$  and thus its inverse exists on p F p p

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Figure S.3: Illustration of the vector bundle. The closed curve in the bottom represents the base manifold M and the figure on the left represents the total space E, where each vertical line represents a fiber. The thickened segment U M represents an open subset of the manifold M, while is a local trivialization defined on  $^{-1}(U)$  that is highlighted in gray in the total space.

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The map in the above definition is called a **local trivialization**. As graphically illustrated in Figure S.3, a vector bundle locally resembles the product space  $U \times \mathbb{R}^k$  for some integer k. A function V defined on M is called a **section** of the vector bundle if  $V(p) = {}^{-1}(p)$  for all p = M. As previously mentioned, the union of all tangent spaces of a manifold, called the **tangent bundle** of the manifold, is a prominent example of the tangent the tangent space at each point is a fiber. In particular, a section of a tangent andle is a section of a manifold.

For a smooth particular  $\mathcal{T}_p$  and  $\mathcal{T}_p$   $\mathcal{T}_p$   $\mathcal{M}$ , the covariant derivative of f at p along is direction of the particular defined.

$$(f)(p) = (f)'(0) = \lim_{t \to 0} \frac{f((t)) - f(p)}{t},$$

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able curve with that (0) = p and '(0) = v. For a smooth vector field p. Let (M) denote the collection of smooth real-valued functions  $\infty(M)$  a smooth vector field U, fU denotes a smooth vector field M pure. Let (E) be the collection of smooth sections.

v UV of the second derivative or the curve. Note that the definition of second derivative involves di erentiating the first derivative. To generalize the concept of acceleration to manifold-valued curves, we then need to di erentiate the velocity — represented by tangent vectors — of the curve, and this involves the concept of connection.

**Definition S.12** (Connection). A connection in a vector bundle E is a map (E) × (E) (E), with  $_{225}$  (V, U)  $_{V}$  U, that satisfies the following properties:

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•  $_V(fU) = f_VU + (_Vf)U$  for  $f_C^{\infty}(\mathbb{M})$ .

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In the above, the value of  $_{V}U$  at p depends on V only through its value at p (Proposition 4.5, Lee, 2018). This observation leads to the definition of covariant derivative of a vector field at p along  $v = T_pM$ . Consequently, the expression  $_{V}U$  is sensible for  $v = T_pM$ , and is called the **covariant derivative** of U at p along the tangent vector v.

Let  $f_{U,V}(p) = U(p), V(p)_p$ . For a connection on the tangent bundle of M, we say is compatible with the metric on M if  $_vf_{U,V} = _vU, V_p + U, _vV_p$  for all smooth vector fields U and V, each p = Mand each tangent vector  $v = T_pM$ . For vector fields U and V, we use [U, V] to denote a new vector field such that  $_{[U,V]}f = _{U=V}f - _{V=U}f$  for all  $f = C^{\infty}(M)$ . Similarly, for  $u, v = T_pM$ , [u, v] denotes the tangent at p such that  $_{[u,v]}f = _{u=v}f - _{V=u}f$  for all  $f = C^{\infty}(M)$ . A connection is torsion-free if  $_{U}V - _{v}U = [U, V]$ for all smooth vector fields U, V. For a Riemannian manifold, there exists a unique connection is both torsionfree and compatible with the Riemannian metric. Such connection is called the **Levi–Civita** connection

and deemed the canonical connection in the tangent bundle.

To identify di erent fibers, one can introduce a parallel transport  $\mathscr{P}$  on a vector bundle along a curve on the base manifold. Such parallel transport must satisfy the following axioms: 1)  $\mathscr{P}_p^p$  is the identity map on  ${}^{-1}(p)$  for all  $p \in M$ , 2)  $\mathscr{P}_{(u)}^{(t)} \mathscr{P}_{(s)}^{(u)} = \mathscr{P}_{(s)}^{(t)}$ , and 3) the dependence of  $\mathscr{P}$  on , s and t are smooth. An example is the vector bundle and the parallel transport constructed in Section 2.4. For a tangent bundle, such parallel transport can be induced by a connection.

**Definition S.13** (Parallel transport). Let be a connection in the tangent bundle of a Riemannian manifold  $\mathbb{M}$ . A smooth vector field U is parallel along  $I = \mathbb{M}$  (with respect to ) if  $_{r(t)}U = 0$  for all t = I. The parallel transport of  $v = T_p\mathbb{M}$  along with p = (0) is U((t)) for the unique smooth vector field U along such that U is parallel along and U(0) = v.

Unlike Euclidean spaces, manifolds are often not flat and exhibit curvature that measures the degree of deviation from being flat. For smooth vector fields U, V, W, we define the map R(U, V, W) = U V W - V W V - UW - [U,V]W. It turns out that the value of R(U, V, W) at p depends only on the values of U, V, W at p, and therefore it is sensible to write R(u, v, w) for tangent vectors at the same point.

Definition S.14 (Sectional curvature). The sectional curvature at p is a real-valued function on  $T_p M \times T_p M$ defined for u, v  $T_p M$  by  $\Re(u, v) = R(u, v, v)$ ,  $u_p (u, u_p v, v_p - u, v_p^2)$ .

Note that sectional curvature is invariant to the length of tangent vectors u and v. We say the sectional curvature of a Riemannian manifold M is upper (lower, resp.) bounded by if  $\mathfrak{K}(u, v)$  ( $\mathfrak{K}(u, v)$ , resp.) for all p M and u, v  $T_pM$ .

# <sup>200</sup> S.2 Asymptotic distribution of the covariance estimator

In this section, we provide a weak convergence result for the estimated covariance under the assumption  $\hat{\mu} = \mu$  that is also adopted in Zhang and Wang (2016) for simplification. With the consistency property of  $\hat{\mu}$ , the derived asymptotic normality under the assumption  $\hat{\mu} = \mu$  may approximate the reality well when sample size is su ciently large. To drop this assumption, a detailed analysis on the asymptotic normality of  $\hat{\mu}$  seems needed. However, this turns out to be very challenging in the context of Riemannian data due to the curvature e ect. Since the focus of this paper is the construction of the covariance vector bundle framework rather than the mean estimation by local linear smoothing, we decide to leave it for future study.

For  $v_s$   $T_{\mu(s)} \land M$  and  $v_t$   $T_{\mu(t)} \land M$ , let  $v_{s,v_t} = Cv_s$ ,  $v_t \mu(t)$  and  $\hat{v}_{s,v_t} = \mathscr{P}^{(\mu(s),\mu(t))}_{(\hat{\mu}(s),\hat{\mu}(t))}\hat{C}(s,t)v_s$ ,  $v_t \mu(t)$ . It is seen that each pair  $(v_s, v_t)$  defines a linear functional on  $\mathbb{L}(\mu(s), \mu(t))$ . Then  $\mathscr{P}^{(\mu(s),\mu(t))}_{(\hat{\mu}(s),\hat{\mu}(t))}\hat{C}(s,t)$  is weakly

<sup>270</sup> convergent to C(s, t) if we can show that  $\hat{v}_{s,V_t}$  weakly converges to  $v_{s,V_t}$  for all  $(v_s, v_t)$  according to the Cramér–Wold device. Below we demonstrate this for the random design; similar result can be proved for the hybrid design, and for deterministic design by utilizing the concept of design densities (Sacks and Ylvisaker, 1970).

To state the result, let f be the probability density of  $T_{11}$ , and  $K^2 = K^2(u)du$ . Define

$$V_{1}(s, t, v_{s}, v_{t}) = \text{Var}\{ \text{Log}_{\mu(T_{1})} X(T_{1}), v_{s} \text{Log}_{\mu(T_{2})} X(T_{2}), v_{t} \quad T_{1} = s, T_{2} = t \},$$

$$V_{2}(s, t, v_{s}, v_{t}) = \text{Cov}\{ \text{Log}_{\mu(T_{1})} X(T_{1}), v_{s} \text{Log}_{\mu(T_{2})} X(T_{2}), v_{t} , \text{Log}_{\mu(T_{1})} X(T_{1}), v_{s} \text{Log}_{\mu(T_{3})} X(T_{3}), v_{t}$$

$$T_{1} = s, T_{2} = t, T_{3} = t \},$$

$$V_{3}(s, t, v_{s}, v_{t}) = \text{Cov}\{ \text{Log}_{\mu(T_{1})} X(T_{1}), v_{s} \text{Log}_{\mu(T_{2})} X(T_{2}), v_{t} , \text{Log}_{\mu(T_{3})} X(T_{3}), v_{s} \text{Log}_{\mu(T_{4})} X(T_{4}), v_{t}$$

$$T_{1} = s, T_{2} = t, T_{3} = s, T_{4} = t \}.$$

**Theorem S.2.1.** Suppose that Assumptions 2.1, 2.2, 3.1, 4.1, 4.4, 4.5 and 4.6 hold. In addition, assume  $h_c^{275} = h_c = 0$ ,  $nm^2 h_c^2 = m^3 h_c^3 = n$ , and  $mh_c = c$  for some constant c = [0, ]. Then for s, t = T and  $v_s = T_{\mu(s)} M$  and  $v_t = T_{\mu(t)} M$ ,

$$C^{-1/2} v_{s,v_t} - v_{s,v_t} - b(h_c) + o_P(h_c^2) N(0,1),$$
(S.1)

where  $b(h) = \frac{1}{2}h^2$   $u^2 K(u) du - \frac{2}{s^2}(s, t) + \frac{2}{t^2}(s, t)$  and

$$_{\mathcal{C}} = \left\{1 + 1_{S=t}\right\} - \frac{\|K\|^4}{nm(m-1)h_c^2} \frac{V_1(s,t,v_s,v_t)}{f(s)f(t)} + \frac{\|K\|^2}{n(m-1)h_c} \frac{f(s)V_2(t,s,v_t,v_s) + f(t)V_2(s,t,v_s,v_t)}{f(s)f(t)} + \frac{(m-2)(m-3)V_3(s,t,v_s,v_t)}{nm(m-1)}$$

The proof for the above theorem follows from Zhang and Wang (2016) once one realizes that  $_{v_s,v_t}$  is the estimated covariance based on the raw covariance  $\text{Log}_{\mu(T_{ij})}X(T_{ij}), v_s \quad \text{Log}_{\mu(T_{ik})}X(T_{ik}), v_t$ . From the theorem we then observe the same phase transition as that in Zhang and Wang (2016) in the following corollary.

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Corollary S.2.1. Assume the conditions of Theorem S.2.1.

(a) When  $m n^{1/4}$ , with  $h_c n^{-1/4}$  and  $mh_c$ , one has

$$\overline{n} \, \hat{v}_{s,v_t} - v_{s,v_t} \, \stackrel{D}{=} \, N \, 0, \, V_1(s, t, v_s, v_t)$$

(b) When  $m n^{1/4} = c_{\circ}$ , with  $h_{c} = c_{*} n^{-1/4}$ , one has

$$\overline{n}_{V_s,V_t} - V_{v_s,V_t} - b(h) \stackrel{D}{=} N(0, *)$$

where

$$* = \{1 + 1_{s=t}\} \quad \frac{K^4}{c_o^2 c_*^2} \frac{V_1(s, t, v_s, v_t)}{f(s)f(t)} + \frac{K^2}{c_o c_*} \frac{f(s)V_2(t, s, v_t, v_s) + f(t)V_2(s, t, v_s, v_t)}{f(s)f(t)} + V_3(s, t, v_s, v_t)$$

(c) When  $m n^{1/4}$ , with  $h_c n^{-1/6} m^{-1/3}$ , one has

$$n^{1/3}m^{2/3} \hat{v}_{s,v_t} - v_{s,v_t} - b(h) \stackrel{D}{\longrightarrow} N \ 0, \{1+1_{s=t}\} \ K \ \frac{4V_1(s,t,v_s,v_t)}{f(s)f(t)}$$

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## S.3 Proofs of Main Results

*Proof of Lemma 2.1.* We prove this result for any fixed  $t \, \text{T}$ . First, we show  $\mathbb{E}\{\log_{\mu(t)}X(t)\}=0$ . Suppose that is a geodesic emanating from  $\mu(t)$  with velocity  $v \, T_{\mu(t)} \mathbb{M}$  and  $v_{\mu(t)}=1$ . According to Proposition 2.10 of Oller and Corcuera (1995), one has

$$\frac{d}{ds}F((s), t) = \mathbf{E}\{2d_{\mathcal{M}}(X(t), \mu(t))\cos v, \log_{\mu(t)}X(t)\}$$
  
=2E{  $\log_{\mu(t)}X(t)\cos v, \log_{\mu(t)}X(t)$  }.

Since F(p, t) reaches the minimum at  $p = \mu(t)$ , we have  $\frac{d}{ds}F((s), t) = 0$  for any  $v = T_{\mu(t)}M$ . As  $\log_{\mu(t)}X(t) \cos v, \log_{\mu(t)}X(t)$  is the projection of  $\log_{\mu(t)}X(t)$  onto  $v, \mathbb{E}\{\log_{\mu(t)}X(t) \cos v, \log_{\mu(t)}X(t)\} = 0$ 

0 for all  $v = T_{\mu(t)} \\ M$  then implies  $E\{Log_{\mu(t)} \\ X(t)\} = 0$ . According to the definition of Y(t), it holds that  $E\{Log_{\mu(t)} \\ Y(t)\} = 0$ . Similarly, it can be shown that the derivative of  $F^*(, t) = E\{d^2_{\mathcal{M}}(Y(t), )\}$  vanishes at  $\mu(t)$ . With Assumption 2.1, this implies that  $\mu(t)$  is the unique minimum of  $F^*(, t)$  and thus is the Fréchet mean of Y(t).

*Proof of Theorem 2.1.* The mean continuity and joint measurability ensure that  $\text{Log}_{\mu}X$  is a random element in  $\mathscr{T}(\mu)$ . According to the definition of  $\mathbb{C}(s, t)$ , for any  $u, v \quad \mathscr{T}(\mu)$ ,

$$\tau^{C(s, \ )} u(s) ds, v \mu = \tau \tau^{C(s, \ t)} u(s), v(t) \mu(t) ds dt$$
  
=  $\tau \tau^{E\{ \log_{\mu(s)} X(s), u(s) \mu(s) \log_{\mu(t)} X(t), v(t) \mu(t) \} ds dt}$   
=  $E \tau^{Log_{\mu(s)} X(s), u(s) \mu(s)} ds \tau^{Log_{\mu(t)} X(t), v(t) \mu(t)} dt$   
=  $E( \log_{\mu} X, u \mu \log_{\mu} X, v \mu) = Cu, v \mu,$ 

which implies that  $(\mathbf{C}u)(t) = \mathcal{T}C(s, t)u(s)ds$ .

for M gives rise to the same smooth structure on  $\mathbb{L}$ .

Proof of Theorem 2.2. To see that { $\begin{pmatrix} -1 \\ U \\ \times U \end{pmatrix}$ , , ) ( , )  $J^2$ } is a smooth atlas, it is su cient to check the transition maps. Suppose that  $(p, q, \stackrel{d}{_{j,k=1}}v_{jk}B_{,j}(p) B_{,k}(q)) \stackrel{-1 \\ U \\ \times U \end{pmatrix} \stackrel{-1 \\ (U \\ \times U ) \stackrel{-1 \\ (U \\ \times U \stackrel{-1 \\ (U \\ \times U \stackrel{-1 \\ (U \\ \times U \stackrel{-1 \\ (U \\ (U \\ \times U ) \stackrel{-1 \\ (U \\ \times U \stackrel{-1 \\ (U \\ (U \\ \times U \stackrel{-1 \\ (U \\ (U \\ (U \\ (U \\ (U \\ (U \\ ($ 

According to the vector bundle construction lemma (Lemma 5.5, Lee, 2002), it is sull cient to check that when  $U = (U \times U)$   $(U^{-} \times U^{-})$  for some indices , , , , , the composite map  $\int_{-1}^{-1} from U \times \mathbb{R}^{d^{2}}$ to itself has the form  $\int_{-1}^{-1} = (p, q, \bigcup (p, q)v)$  for a smooth map  $\bigcup U$   $GL(d^{2}, \mathbb{R})$ , where  $GL(d^{2}, \mathbb{R})$  is the collection of invertible real  $d^{2} \times d^{2}$  matrices. From above discussion, we have  $\bigcup (p, q) = \bigcup^{-1}(p) \quad \bigcup^{-1}(q)$  is smooth in (p, q). In addition,  $\bigcup (p, q) \quad GL(d^{2}, \mathbb{R})$  since both  $\bigcup^{-1}(p)$  and  $\bigcup^{-1}(q)$  are invertible and so is their Kronecker product. Note that the vector bundle construction lemma also asserts that any compatible atlas

*Proof of Theorem 2.3.* One can show that the parallel transport defined in (5) is a genuine parallel transport satisfying the property of Definition A.54 of Rodrigues and Capelas de Oliveira (2007) on the vector bundle.

Then the conclusion directly follows from Definitions A.55 and A.57 of Rodrigues and Capelas de Oliveira (2007) and the remarks right below them.

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Proof of Theorem 2.4. We first show that the definition (7) is invariant to the choice of orthonormal bases. To this end, fix an orthonormal basis in  $T_q M$ , and suppose that  $\{\tilde{e}_1, \ldots, \tilde{e}_d\}$  is another orthonormal basis in  $T_p M$  and is related to  $\{e_1, \ldots, e_d\}$  by a  $d \times d$  unitary matrix **O**. Let  $A_1$  and  $A_2$  be the respective matrix representation of  $L_1$  and  $L_2$  under the basis  $\{e_1, \ldots, e_d\}$ . Then their matrix representation under the basis  $\{\tilde{e}_1, \ldots, \tilde{e}_d\}$  is  $\tilde{A}_1 = \mathbf{O}A_1$  and  $\tilde{A}_2 = \mathbf{O}A_2$ , respectively. The inner product  $G_{p,q}(L_1, L_2)$  is then calculated by  $\operatorname{tr}(\tilde{A}_1^T \tilde{A}_2) = \operatorname{tr}(A_1^T \mathbf{O}^T \mathbf{O}A_2) = \operatorname{tr}(A_1^T A_2)$ , which shows that  $G_{p,q}(L_1, L_2)$  is invariant to the choice of bases in  $T_p M$ . Its invariance to the choice of bases in  $T_q M$  can be proved in a similar fashion.

The smoothness of *G* can be established by an argument similar to the one leading to Theorem 2.2 in conjunction with smoothness of the trace of matrices. To see that the parallel transport (5) preserves the bundle metric and thus defines isometries among fibers of  $\mathbb{L}$ , i.e., for any  $L_1, L_2 = \mathbb{L}(p_1, q_1)$ ,

$$G_{(p_1,q_1)}(L_1,L_2) = G_{(p_2,q_2)}(\mathscr{P}_{(p_1,q_1)}^{(p_2,q_2)}L_1,\mathscr{P}_{(p_1,q_1)}^{(p_2,q_2)}L_2),$$

suppose that  $\{e_1, \ldots, e_d\}$  is an orthogonal basis of  $\mathcal{T}_{p_1} \mathcal{M}$ . Then  $\{\mathsf{P}_{p_1}^{p_2} e_1, \ldots, \mathsf{P}_{p_1}^{p_2} e_d\}$  is an orthogonal basis of  $\mathcal{T}_{p_2} \mathcal{M}$ . This further implies that

$$G_{(p_{2},q_{2})}(\mathscr{P}_{(p_{1},q_{1})}^{(p_{2},q_{2})}L_{1},\mathscr{P}_{(p_{1},q_{1})}^{(p_{2},q_{2})}L_{2}) = \overset{d}{\underset{k=1}{\overset{d}{\underset{k=1}{\overset{(\mathscr{P}_{(p_{2},q_{2})}{\underset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}{\overset{(p_{2},q_{2})}{\overset{(p_{2},q_{2})}{\overset{(p_{2},q_{2})}{\overset{(p_{2},q_{2})}}}}, (\mathscr{P}_{(p_{1},q_{1})}^{(p_{2},q_{2})}L_{2})(\mathbb{P}_{p_{1}}^{p_{2}}e_{k}) |_{q_{2}}$$
$$= \overset{d}{\underset{k=1}{\overset{d}{\underset{k=1}{\overset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}{\overset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}{\overset{(p_{2},q_{2})}{\overset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}{\overset{(p_{2},q_{2})}{\overset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}}}} + \overset{d}{\underset{k=1}{\overset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}{\overset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}{\overset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}{\overset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}{\overset{(p_{1},q_{1})}{\overset{(p_{2},q_{2})}{\overset{(p_{1},q_{1})}{\overset{(p_{1},q_{1})}{\overset{(p_{1},q_{1})}{\overset{(p_{1},q_{1})}{\overset{(p_{1},q_{1})}{\overset{(p_{1},q_{1})}}}}} + \overset{d}{\underset{k=1}{\overset{(p_{1},q_{1})}{\overset{(p_{1},q$$

which completes the proof.

*Proof of Proposition 3.1.* Suppose that  $(\tilde{B}_{ij,1}, ..., \tilde{B}_{ij,d})$  is another orthonormal basis for  $T_{\hat{\mu}(T_{ij})}M$ , and  $O_{ij}$  is the unitary matrix relating  $(B_{ij,1}, ..., B_{ij,d})$  to  $(\tilde{B}_{ij,1}, ..., \tilde{B}_{ij,d})$ . Then the coe cient vectors  $\tilde{z}_{ij}$  and  $\tilde{g}_{k,ij}$  of  $\text{Log}_{\hat{\mu}(T_{ij})}Y_{ij}$  and  $\hat{k}(T_{ij})$  under the basis  $(\tilde{B}_{ij,1}, ..., \tilde{B}_{ij,d})$  are linked to  $z_{ij}$  and  $g_{k,ij}$  by  $\tilde{z}_{ij} = O_{ij}z_{ij}$  and  $\tilde{g}_{k,ij} = O_{ij}g_{k,ij}$ , respectively. Similarly,  $\tilde{C}_{i,j,l}$  is linked to  $C_{i,j,l}$  by  $\tilde{C}_{i,j,l} = O_{ij}C_{i,j,l}O_{il}^{\mathsf{T}}$ . More concisely, if we put



then  $\tilde{z}_i = \mathbf{O}_i z_i$ ,  $\tilde{g}_{k,i} = \mathbf{O}_i g_{k,i}$  and  $\tilde{i} = \mathbf{O}_i O_i^{\mathsf{T}}$ , which are the counterpart of  $z_i$ ,  $g_{k,i}$  and i under the bases  $(\tilde{B}_{ij,1}, \ldots, \tilde{B}_{ij,d})$ , respectively. Note that  $\tilde{i}^{\mathsf{T}}_1 = (\mathbf{O}_i O_i^{\mathsf{T}})^{-1} = \mathbf{O}_i^{\mathsf{T}} \tilde{i}^{\mathsf{T}} \mathbf{O}_i^{\mathsf{T}} = \mathbf{O}_i \tilde{i}^{\mathsf{T}} \mathbf{O}_i^{\mathsf{T}}$  are unitary matrices and thus  $\mathbf{O}_i^{\mathsf{T}}_1 = \mathbf{O}_i^{\mathsf{T}}$ . Now we see that  $\tilde{g}_{k,i}^{\mathsf{T}} \tilde{i}^{\mathsf{T}} \tilde{z}_i = g_{k,i}^{\mathsf{T}} \mathbf{O}_i^{\mathsf{T}} \mathbf{O}_i \tilde{i}^{\mathsf{T}} \mathbf{O}_i^{\mathsf{T}} \mathbf{O}_i z_i = g_{k,i}^{\mathsf{T}} \tilde{i}^{\mathsf{T}} z_i$ , which clearly implies that the scores  $\tilde{i}_k$  calculated under the bases  $(\tilde{B}_{ij,1}, \ldots, \tilde{B}_{ij,d})$  is identical to the one computed under the bases  $(B_{ij,1}, \ldots, B_{ij,d})$ .

Proof of Lemma 4.1. Notice that

$$\begin{aligned} & P_{q_1}^{p_1} P_{q_2}^{q_1} \log_{q_2} y - P_{p_2}^{p_1} \log_{p_2} y_{p_1} \\ &= P_{p_2}^{q_2} P_{p_1}^{p_2} P_{q_1}^{p_1} P_{q_2}^{q_1} \log_{q_2} y - P_{p_2}^{q_2} \log_{p_2} y_{q_2} \\ & P_{p_2}^{q_2} P_{p_1}^{p_2} P_{q_1}^{p_1} P_{q_2}^{q_1} \log_{q_2} y - \log_{q_2} y_{q_2} + \log_{q_2} y - P_{p_2}^{q_2} \log_{p_2} y_{q_2} \end{aligned}$$

where the equality follows from the fact that parallel transport preserves the inner product. Note that the operator  $P_{p_2}^{q_2}P_{p_1}^{p_2}P_{q_1}^{p_1}P_{q_2}^{q_1}$  moves a tangent vector parallelly along a geodesic quadrilateral defined by the the points  $p_1, p_2, q_1, q_2$ . The holonomy theory (Eq (6), Nichols et al., 2016) and the compactness of G suggests that there exists a constant  $c_1 > 0$  depending only on G, such that for any  $v = T_{q_2} M$  with  $v_{q_2} = \text{diam}(G)$ ,

$$\begin{split} & \mathsf{P}_{p_2}^{q_2} \mathsf{P}_{p_1}^{p_2} \mathsf{P}_{q_2}^{p_1} v - v_{q_2} & c_1 \ \mathsf{Log}_{q_2} p_2 \ q_2 = c_1 d_{\mathcal{M}}(p_2, q_2), \\ & \mathsf{P}_{p_1}^{q_2} \mathsf{P}_{q_1}^{p_1} \mathsf{P}_{q_2}^{q_1} v - v_{q_2} & c_1 \ \mathsf{Log}_{q_1} p_1 \ q_2 = c_1 d_{\mathcal{M}}(p_1, q_1), \end{split}$$

which further imply that

$$\begin{split} \mathsf{P}_{p_{2}}^{q_{2}} \mathsf{P}_{p_{1}}^{p_{2}} \mathsf{P}_{q_{1}}^{p_{1}} \mathsf{P}_{q_{2}}^{q_{1}} \mathsf{V} - \mathsf{V}_{q_{2}} = & \left(\mathsf{P}_{p_{2}}^{q_{2}} \mathsf{P}_{p_{1}}^{p_{2}} \mathsf{P}_{q_{2}}^{p_{1}}\right) \left(\mathsf{P}_{q_{1}}^{q_{2}} \mathsf{P}_{q_{1}}^{p_{1}} \mathsf{P}_{q_{2}}^{q_{1}}\right) \mathsf{V} - \mathsf{V}_{q_{2}} \\ & \left(\mathsf{P}_{p_{2}}^{q_{2}} \mathsf{P}_{p_{1}}^{p_{2}} \mathsf{P}_{q_{2}}^{p_{1}}\right) \left(\mathsf{P}_{q_{1}}^{q_{2}} \mathsf{P}_{q_{1}}^{p_{1}} \mathsf{P}_{q_{2}}^{q_{1}}\right) \mathsf{V} - \left(\mathsf{P}_{p_{1}}^{q_{2}} \mathsf{P}_{q_{1}}^{p_{1}} \mathsf{P}_{q_{2}}^{q_{1}}\right) \mathsf{V}_{q_{2}} + & \left(\mathsf{P}_{p_{1}}^{q_{2}} \mathsf{P}_{q_{1}}^{p_{1}} \mathsf{P}_{q_{2}}^{q_{1}}\right) \mathsf{V} - \mathsf{V}_{q_{2}} \\ & c_{1} \left(d_{\mathcal{M}}(p_{2}, q_{2}) + d_{\mathcal{M}}(p_{1}, q_{1})\right). \end{split}$$

According to Theorem 3 in Pennec (2019), we have

$$\log_{q_2} y - P_{p_2}^{q_2} \log_{p_2} y_{q_2} \quad c_2 \log_{q_2} p_2 \quad q_2 \quad c_2 d_{\mathcal{M}}(p_2, q_2)$$

for some constant  $c_2 > 0$  depending only on G. The proof is then completed by taking  $c = c_1 + c_2$ .

Proof of Propositions 4.1 and 4.2. Simple computation shows that

$$\begin{split} & \hat{Q}_{n}(y, \cdot) - F^{*}(y, \cdot) \\ &= \frac{\hat{u}_{2}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, \cdot) \\ &- \frac{\hat{u}_{1}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) (T_{ij} - \cdot) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, \cdot) \\ &= \frac{\hat{u}_{2}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, \cdot) - F^{*}(y, \cdot) (T_{ij} - \cdot) \\ &- \frac{\hat{u}_{1}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) (T_{ij} - \cdot) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, \cdot) - F^{*}(y, \cdot) (T_{ij} - \cdot) \\ &- \frac{\hat{u}_{1}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) (T_{ij} - \cdot) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, \cdot) - F^{*}(y, \cdot) (T_{ij} - \cdot) \\ &- \frac{\hat{u}_{1}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) (T_{ij} - \cdot) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, \cdot) - F^{*}(y, \cdot) (T_{ij} - \cdot) \\ &- \frac{\hat{u}_{1}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) (T_{ij} - \cdot) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, \cdot) - F^{*}(y, \cdot) (T_{ij} - \cdot) \\ &- \frac{\hat{u}_{1}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) (T_{ij} - \cdot) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, \cdot) - F^{*}(y, \cdot) (T_{ij} - \cdot) \\ &- \frac{\hat{u}_{1}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) (T_{ij} - \cdot) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, \cdot) - F^{*}(y, \cdot) (T_{ij} - \cdot) \\ &- \frac{\hat{u}_{1}(\cdot)}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{nm} \int_{ij} K_{h\mu}(T_{ij} - \cdot) (T_{ij} - \cdot) \int_{ij} \frac{1}{\hat{r}_{0}^{2}(\cdot)} \frac{1}{\hat{r}_{$$

Below we focus on the fist term, noting that the second term can be analyzed in a similar way.

Define

$$U = \frac{1}{nm} K_{h_{\mu}}(T_{ij} - ) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, ) - F^{*}(y, )(T_{ij} - )$$

Then, according to either Lemma S.4.1 or Lemma S.4.3, the rate of the first term depends on the rate of U. By Taylor expansion of  $F^*(y, T_{ij})$  at and Assumption 4.5(a), we have

$$\sup_{\epsilon \in B(t;h)} \mathbf{E} U = \sup_{\epsilon \in B(t;h)} \mathbf{E} \frac{1}{nm} K_{h_{\mu}}(T_{ij} - ) F^{*}(y, T_{ij}) - F^{*}(y, ) - F^{*}(y, )(T_{ij} - )$$
$$= \sup_{\epsilon \in B(t;h)} \mathbf{E} \frac{1}{nm} K_{h_{\mu}}(T_{ij} - ) \times O(h_{\mu}^{2}) = O(h_{\mu}^{2}).$$

For the random and hybrid designs, define the envelop function

$$H = \frac{2\text{diam}(K)^2}{m} \prod_{j=1}^{m} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{1j} - ).$$

According to Lemma S.4.1(a), we have  $E(H^2) = O(1 + \frac{1}{mh_{\mu}})$  and thus

$$\sup_{e \in B(t;h)} U - \mathbf{E}U = O_p \qquad \overline{\frac{1}{n} + \frac{1}{nmh_{\mu}}}$$

according to Theorems 2.7.11 and 2.14.2 of van der Vaart and Wellner (1996). Lemma S.4.6 asserts that the last equation also holds for a deterministic design. With Lemma S.4.1 we deduce that

$$\sup_{\epsilon \in B(t;h)} \hat{Q}_n(y, ) - F^*(y, ) = O_p h_{\mu}^2 + \frac{1}{n} + \frac{1}{nmh_{\mu}}$$

A similar argument leads to

$$\sup_{\substack{d_{\mathcal{M}}(y_1,y_2)<\delta\\\in\mathcal{B}(t,h)}} \hat{Q}_n(y_1, \cdot) - \hat{Q}_n(y_2, \cdot) - F^*(y_1, \cdot) - F^*(y_2, \cdot) = O_p \quad h_{\mu}^2 + \frac{1}{n} + \frac{1}{nmh_{\mu}} \quad . \tag{S.2}$$

for any  $y_1, y_2 \in K$  and > 0. Following from the argument in the proof of Lemma 2 in Petersen and Müller (2019), one can verify that for any > 0

$$\lim_{\to 0} \limsup_{n \to \infty} \Pr \sup_{d_{\mathcal{M}}(y_1, y_2) < , \in B(t;h)} \hat{Q}_n(y_1, \cdot) - \hat{Q}_n(y_2, \cdot) - F^*(y_1, \cdot) - F^*(y_2, \cdot) > = 0,$$

and further

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$$\sup_{\epsilon B(t;h)} d_{\mathcal{M}}(\mu(\cdot), \hat{\mu}(\cdot)) = o_{\rho}(1)$$
(S.3)

given Assumption 4.5(b).

To derive the rate we apply (S.2) with  $y_1 = y$  and  $y_2 = \mu($ ) to obtain

$$\sup_{\substack{d_{\mathcal{M}}(y,\mu(\tau))<\delta\\\in\mathcal{B}(t,h)}} \hat{Q}_{n}(y, ) - \hat{Q}_{n}(\mu(), ) - F^{*}(y, ) - F^{*}(\mu(), ) = O_{p} \quad h_{\mu}^{2} + \frac{1}{n} + \frac{1}{nmh_{\mu}} \quad (S.4)$$

By (S.3), the the event  $\{d_{\mathcal{M}}(\hat{\mu}(\cdot), \mu(\cdot)) < 1\}$  occurs with probability tending to one. On this event, according to Assumption 4.5(c), we have

$$F^*(\hat{\mu}(),) - F^*(\mu(),) - C_1 d_{\mathcal{M}}(\hat{\mu}(),\mu())^2 = 0.$$

Since  $\hat{\mu}()$  is the minimizer of  $\hat{Q}_n(y, )$ , we have  $\hat{Q}_n(\mu(), ) - \hat{Q}_n(\hat{\mu}(), )$  0 and the following inequality on the event  $\{d_{\mathcal{M}}(\hat{\mu}(), \mu()) < 1\}$ ,

$$F^{*}(\hat{\mu}(),) - F^{*}(\mu(),) - \hat{Q}_{n}(\hat{\mu}(),) - \hat{Q}_{n}(\mu(),) - \hat{Q}$$

Let  $a_n = h_{\mu}^2 + \overline{n^{-1} + (nmh_{\mu})^{-1}}$ . Below we fix an arbitrary > 0, and find M > 0 accordingly to satisfy

$$\Pr \sup_{\in B(t;h)} d_{\mathcal{M}}(\hat{\mu}(),\mu()) > Ma_n \quad .$$

To this end, for R > 0 to be determined later, let

$$B_{R}(\) = \sup_{\substack{d_{\mathcal{M}}(y,\mu(\tau)) < \delta \\ \in B(t;h)}} \hat{Q}_{n}(y,\) - \hat{Q}_{n}(\mu(\),\) - F^{*}(y,\) - F^{*}(\mu(\),\) \qquad R \quad h_{\mu}^{2} + \frac{1}{n} + \frac{1}{nmh_{\mu}} \quad ,$$

$$B_{j} = \{2^{j}Ma_{n} \quad \sup_{\epsilon B(t;h)} d_{\mathcal{M}}(\hat{\mu}(\),\mu(\)) \quad 2^{j+1}Ma_{n}\},$$

$$B_{C} = \{ \sup_{\epsilon B(t;h)} d_{\mathcal{M}}(\hat{\mu}(\),\mu(\)) > \frac{1}{2} \ 1 \}.$$

Let  $j_0 = 0$  be an integer satisfying  $\frac{1}{2} + 2^{j_0+1}Ma_n = 1$ . We then have

$$\Pr \sup_{\boldsymbol{\epsilon} B(t;h)} d_{\mathcal{M}}(\hat{\mu}(\cdot), \mu(\cdot)) > Ma_{n} \int_{j=0}^{j_{0}} \Pr\{B_{j} \mid B_{R}(2_{-1})\} + \Pr\{B_{C} \mid B_{R}(2_{-1})\} + \Pr\{B_{R}(2_{-1})\} + \Pr\{B_{R}(2_{-1})\} + \Pr\{B_{R}(2_{-1})\} + \Pr\{B_{R}(2_{-1})\} + \Pr\{B_{R}(2_{-1})\} = 0$$

deterministic design, we can show that

$$\sup_{\epsilon \mathcal{T}, y \in \mathcal{K}} \hat{Q}_n(y, \cdot) - F^*(y, \cdot) = O_p \cdot h_\mu^2 + \frac{1}{n} + \frac{1}{nmh_\mu} \cdot (\log n)^{1/2}.$$

Together with (S.5), this proves Proposition 4.1.

Proof of Theorems 4.1. We divide the proof into three steps. In the first step, we identify two key terms that determine the convergence rate. In the second step and third step, we address the terms separately. Note that  $v_1 = -v_n = 1 \{nm(m-1)\}$ .

Step 1. Define =  $\hat{\mu}$  to simplify notation in the sequel. Since the parallel transport  $P_{(s),(t)}^{(\mu(s),\mu(t))}$  preserves the fiber metric according to Theorem 2.4,

Combining (S.6) and (S.8), we deduce that

$$\begin{aligned} \mathscr{P}_{((s), (t))}^{(\mu(s), \mu(t))} \hat{\mathbb{C}}(s, t) - \mathbb{C}(s, t) \\ &= \frac{(S_{20}S_{02} - S_{11}^2)[R_{00,1} + R_{00,2} - {}_{s}\mathbb{C}(s, t)h_{\mathcal{C}}S_{10} - {}_{t}\mathbb{C}(s, t)h_{\mathcal{C}}S_{01}]}{(S_{20}S_{02} - S_{11}^2)S_{00} - (S_{10}S_{02} - S_{01}S_{11})S_{10} + (S_{10}S_{11} - S_{01}S_{20})S_{01}} \\ &- \frac{(S_{10}S_{02} - S_{01}S_{11})[R_{10,1} + R_{10,2} - {}_{s}\mathbb{C}(s, t)h_{\mathcal{C}}S_{20} - {}_{t}\mathbb{C}(s, t)h_{\mathcal{C}}S_{11}]}{(S_{20}S_{02} - S_{11}^2)S_{00} - (S_{10}S_{02} - S_{01}S_{11})S_{10} + (S_{10}S_{11} - S_{01}S_{20})S_{01}} \\ &+ \frac{(S_{10}S_{11} - S_{01}S_{20})[R_{01,1} + R_{01,2} - {}_{s}\mathbb{C}(s, t)h_{\mathcal{C}}S_{11} - {}_{t}\mathbb{C}(s, t)h_{\mathcal{C}}S_{02}]}{(S_{20}S_{02} - S_{11}^2)S_{00} - (S_{10}S_{02} - S_{01}S_{11})S_{10} + (S_{10}S_{11} - S_{01}S_{20})S_{01}}. \end{aligned}$$

In light of Lemmas S.4.2 and S.4.3, the convergence rate of (S.6) depends on  $R_{ab,1}$  and  $R_{ab,2}$  -  ${}_{s}C(s, t)h_{c}S_{a+1,b}$  -  ${}_{t}C(s, t)h_{c}S_{a,b+1}$ .

Step 2. In this step we address  $R_{ab,1}$ . According to the definition of  $\mathscr{P}$  in (5), the first part in Equation (S.8) is

$$\mathbb{P}_{(s)}^{\mu(s)} \mathbb{P}_{(T_{ij})}^{(s)} \text{Log}_{(T_{ij})} Y_{jj} \qquad \mathbb{P}_{(t)}^{\mu(t)} \mathbb{P}_{(T_{ik})}^{(t)} \text{Log}_{(T_{ik})} Y_{jk} - \mathbb{P}_{\mu(T_{ij})}^{\mu(s)} \text{Log}_{\mu(T_{ij})} Y_{jj} \qquad \mathbb{P}_{\mu(T_{ik})}^{\mu(t)} \text{Log}_{\mu(T_{ik})} Y_{jk} .$$

Then according to Assumption 4.4(b) and Lemma 4.1, its rate is

$$\mathscr{P}_{(s), \mu(t)}^{(\mu(s), \mu(t))} \mathscr{P}_{(T_{ij}), (T_{ik})}^{(s), (t)} \hat{C}_{i, jk} - \tilde{C}_{i, jk} = O \qquad \sup_{|s| = s | sh_{\mathcal{C}} \text{ or } |s| - t | sh_{\mathcal{C}}} d_{\mathcal{M}}(s) + \mu(s) = O$$

By Proposition 4.2, we conclude that

$$R_{ab,1} = O_p h_{\mu}^2 + \overline{\frac{1}{n} + \frac{1}{nmh_{\mu}}}$$

Step 3. In this step, we first analyze the term  $R_{00,2} - {}_{s}C(s, t)h_{c}S_{10} - {}_{t}C(s, t)h_{c}S_{01}$  in (S.9), which equals to

$$U = \underset{\substack{i \\ j \neq k}}{i} (T_{ij}, T_{ik}) \tilde{C}_{i,jk} - C(s, t) - _{s}C(s, t)(T_{ij} - s) - _{t}C(s, t)(T_{ik} - t)$$

We start with bounding its mean. Let  $\mathbb{T} = \{T_{ij} \mid i = 1, ..., n, j = 1, ..., m_i\}$  and observe that

$$\mathbf{E} \quad \tilde{\mathbb{C}}_{i,jk} \quad \mathbb{T} = \mathscr{P}^{(\mu(s),\mu(t))}_{(\mu(T_{ij}),\mu(T_{ik}))} \mathbb{C}(T_{ij}, T_{ik})$$

In addition, since C is twice di erentiable and the parallel transport  $\mathscr{P}$  is depicted by a partial di erential equation, we have the following Taylor expansion at (s, t),

$$\mathscr{P}_{(\mu(s),\mu(t))}^{(\mu(s),\mu(t))} C(T_{ij},T_{ik}) = C(s,t) + {}_{s}C(s,t)(T_{ij}-s) + {}_{t}C(s,t)(T_{ik}-t) + O(h_{\mathcal{C}}^{2})$$
(S.10)

for all  $T_{ij}$ ,  $T_{ik}$  such that  $T_{ij} - s < h_c$  and  $T_{ik} - t < h_c$ , where  $O(h_c^2)$  is uniform over all  $T_{ij}$  and  $T_{ik}$  due to Assumption 4.6 and the compactness of K. Then we further deduce that

$$E(U) = E E_{i \ j \neq k} (T_{ij}, T_{ik}) \tilde{C}_{i,jk} - C(s, t) - {}_{s}C(s, t)(T_{ij} - s) - {}_{t}C(s, t)(T_{ik} - t) \mathbb{T}$$
$$= E_{i \ j \neq k} (T_{ij}, T_{ik}) \times O(h_{\mathcal{C}}^{2}) = O(h_{\mathcal{C}}^{2}).$$

For the random and hybrid designs, the i.i.d assumption on trajectories and Lemma S.4.2 imply that

$$\mathbf{E} \ U - \mathbf{E} U \stackrel{2}{}_{G} \frac{1}{n^{2}} \stackrel{n}{}_{i=1} \mathbf{E} \frac{1}{m(m-1)} (T_{ij}, T_{ik}) \stackrel{\tilde{C}_{i,jk}}{=} \mathcal{P}^{(\mu(s),\mu(t))}_{(\mu(T_{ij}),\mu(T_{ik}))} \mathbb{C}(T_{ij}, T_{ik}) \stackrel{2}{=} \frac{4\text{diam}(\mathsf{M})^{4}}{n} \mathbf{E} \frac{1}{m(m-1)} (T_{ij}, T_{ik}) \stackrel{2}{=} O \frac{1}{n} + \frac{1}{nm^{2}h_{C}^{2}} .$$

Lemma S.4.7 asserts that this also holds for the deterministic design. Combining this with  $\mathbf{E}U = O(h_c^2)$ , we deduce that  $\mathbf{E} U - \mathbf{E}U_G^2 = O(n^{-1} + n^{-1}m^{-2}h_c^{-2})$ , and with Markov inequality, further conclude that  $R_{00,2} - {}_{s}\mathbb{C}(s, t)h_cS_{10} - {}_{t}\mathbb{C}(s, t)h_cS_{01} = O_p(h_c^2 + n^{-1/2} + n^{-1/2}m^{-1}h_c^{-1})$ .

Similar arguments can show that the terms  $R_{10,2} = {}_{s}\mathbb{C}(s, t)h_{\mathcal{C}}S_{20} = {}_{t}\mathbb{C}(s, t)h_{\mathcal{C}}S_{11}$  and  $R_{01,2} = {}_{s}\mathbb{C}(s, t)h_{\mathcal{C}}S_{11} - {}_{t}\mathbb{C}(s, t)h_{\mathcal{C}}S_{02}$  in (S.9) are of the same order. The equation (10) is then obtained by inserting the results in Steps 2 and 3 into Step 1.

*Proof of Theorem 4.2.* Similar to the proof of Theorem 4.1, we only need to consider the uniform rate of the term  $R_{00,1} + R_{00,2} - {}_{s}\mathbb{C}(s, t)h_{\mathcal{C}}S_{10} - {}_{t}\mathbb{C}(s, t)h_{\mathcal{C}}S_{01}$  in (S.9).

Due to boundedness of K and Lemma 4.1, we have

$$\sup_{s,t} \mathsf{P}_{(s), \mu(t)}^{(\mu(s), \mu(t))} \mathsf{P}_{(T_{ij}), (T_{ik}))}^{(s), (t)} \hat{C}_{i,jk} - \tilde{C}_{i,jk} G \quad C \sup d_{\mathcal{M}}((), \mu()).$$

Therefore, according to Lemma S.4.2 and Proposition 4.1, we deduce that

$$\sup_{s,t} R_{00,1 \ G} c \sup_{s,t} S_{00} \sup d_{\mathcal{M}}(\ (\ ),\mu(\ ) = O_p h_{\mu}^2 + \frac{\log n}{nmh_{\mu}} + \frac{\log n}{n}$$

for a universal constant c > 0 depending only on K.

The uniform convergence rates of  $R_{00,2} - {}_{s}C(s,t)h_{c}S_{10} - {}_{t}C(s,t)h_{c}S_{01}$  and other similar terms are obtained by arguments similar to those in Theorem 5.2 of Zhang and Wang (2016), except that no truncation argument is needed due to Assumption 4.4(b), and moments of some random quantities are calculated by using the techniques in Lemma S.4.2 for the hybrid design and by using Lemma S.4.7 for the deterministic design.

## S.4 Technical Lemmas

The following lemma is used to establish the convergence rate of the mean estimator under the random or hybrid design.

**Lemma S.4.1** (mean, random). Suppose that Assumptions 2.1, 2.2, 3.1, 4.4, 4.5. Under either Assumption 4.3 if  $h_{\mu} = 0$  and  $nmh_{\mu}$ , then for any t and  $h = O(h_{\mu})$ , we have

- (a)  $\mathbf{E} \frac{1}{m} \int \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{1j} )^2 = O (1 + \frac{1}{mh_{\mu}})^2$
- (b) sup  $_{\epsilon \mathcal{T}} \hat{u}_k() = O_p(h^k_\mu)$  for k = 0, 1, 2;
- (c) inf  $_{\in B(t;h)} \stackrel{2}{_{0}} () \quad h_{\mathcal{U}}^{2}(1+o_{\mathcal{P}}(1)).$

$$\sup_{\epsilon B(t;h)} \mathbf{E}\hat{u}_{k} = \sup_{\epsilon B(t;h)} \frac{1}{nm} \sum_{ij} \mathbf{E}[K_{h_{\mu}}(T_{ij} - )(T_{ij} - )^{k}] = \sup_{\epsilon B(t;h)} \mathbf{E}K_{h_{\mu}}(T_{11} - )(T_{11} - )^{k} = O(h_{\mu}^{k}).$$

Define an envelop function

$$H_{k} = \frac{1}{m} \sup_{j \in B(t;h)} K_{h_{\mu}}(T_{1j} - )(T_{1j} - )^{k}$$

for  $\hat{u}_k$ . Under Assumption 4.1, the second moment of  $H_k$  is

$$\mathbf{E}(H_k^2) = \frac{1}{m^2} \sum_{j_1, j_2} \mathbf{E} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{1j_1} - )(T_{1j_1} - )^k \times \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{1j_2} - )(T_{1j_2} - )^k$$

$$= \frac{1}{m} \mathbf{E} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{1j_1} - )(T_{1j_1} - )^{k/2}$$

$$+ \frac{m-1}{m} \mathbf{E} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{11} - )(T_{11} - )^k \times \mathbf{E} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{12} - )(T_{12} - )^k$$

$$= O h_{\mu}^{2k}(1 + \frac{1}{mh_{\mu}}) .$$

Under Assumption 4.3,

$$\mathbf{E}(H_{k}^{2}) = \frac{1}{m^{2}} \sum_{j_{1},j_{2}} \mathbf{E} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{1j_{1}} - )(T_{1j_{1}} - )^{k} \times \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{1j_{2}} - )(T_{1j_{2}} - )^{k}$$

$$= \frac{1}{m} \mathbf{E} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{1j_{1}} - )(T_{1j_{1}} - )^{k} 2$$

$$+ \frac{m-1}{m} \mathbf{E} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{11} - )(T_{11} - )^{k} \times \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{12} - )(T_{12} - )^{k}$$

$$O \frac{h_{\mu}^{2k-1}}{m} + \mathbf{E} \mathbf{E} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{11} - )(T_{11} - )^{k} S_{11}, S_{12}$$

$$\times \mathbf{E} \sup_{\epsilon B(t;h)} K_{h_{\mu}}(T_{12} - )(T_{12} - )^{k} S_{11}, S_{12}$$

$$O \frac{h_{\mu}^{2k-1}}{m} + O(h_{\mu}^{2k-2}) \mathbf{E} \mathbf{E} \mathbf{1}_{t-S_{11}-O(h_{\mu}) \leq 11 \leq t-S_{11}+O(h_{\mu})} S_{11} \mathbf{E} \mathbf{1}_{t-S_{12}-O(h_{\mu}) \leq 12 \leq t-S_{12}+O(h_{\mu})} S_{12}$$

When  $h_{\mu} = L^{-1}$ ,  $\mathbf{E} = 1_{t-S_{11}-O(h_{\mu}) \leq 11 \leq t-S_{11}+O(h_{\mu})} = S_{11}$  is of order O(hL) when  $S_{11} - t = O(L^{-1})$  and zero otherwise, an similar observation applies to  $\mathbf{E} = 1_{t-S_{12}-O(h_{\mu}) \leq 12 \leq t-S_{12}+O(h_{\mu})} = S_{12}$ . Together, they imply that

$$\mathbf{E} \ \mathbf{E} \ \mathbf{1}_{t-S_{11}-O(h_{\mu}) \leq 11 \leq t-S_{11}+O(h_{\mu})} S_{11} \ \mathbf{E} \ \mathbf{1}_{t-S_{11}-O(h_{\mu}) \leq 11 \leq t-S_{11}+O(h_{\mu})} S_{12}$$
  
=  $O(h_{\mu}^{2}L^{2}) \mathbf{E} \{\mathbf{1}_{|S_{11}-t|=O(L^{-1})}\mathbf{1}_{|S_{12}-t|=O(L^{-1})}\} = O(h_{\mu}^{2}L^{2})O(L^{-2}) = O(h_{\mu}^{2}).$ 

When  $h_{\mu} = L^{-1}$ ,  $\mathbf{E} = 1_{t-S_{11}-O(h_{\mu}) \leq 11 \leq t-S_{11}+O(h_{\mu})} = S_{11}$  is of order O(1) when  $S_{11} - t = O(h_{\mu})$  and zero otherwise, an similar observation applies to  $\mathbf{E} = 1_{t-S_{12}-O(h_{\mu}) \leq 12 \leq t-S_{12}+O(h_{\mu})} = S_{12}$ . Together, they imply that

$$E E 1_{t-S_{11} \le 11 \le t-S_{11}+2h} S_{11} E 1_{t-S_{12} \le 12 \le t-S_{12}+2h} S_{12}$$
  
=  $O(1)E\{1_{|S_{11}-t|=O(h_{\mu})}1_{|S_{12}-t|=O(L^{-1})}\} = O(1)O(h_{\mu}^{2}) = O(h_{\mu}^{2}).$ 

#### S.4 TECHNICAL LEMMAS

In summary, we still have  $\mathbf{E}H_k^2 = O h_\mu^{2k}(1 + \frac{1}{mh_\mu})$  under Assumption 4.3. Part (a) is then verified by taking k = 0 in the above.

Part (b) can be proved by an argument analogous to the proof for Lemma 4 of Zhang and Wang (2016). For part (c), it is seen that  $\hat{}_0() \{ E\hat{u}_0 E\hat{u}_2 - (E\hat{u}_1)^2\}(1 + o_P(1)), \text{ where the } o_P(1) \text{ component is uniform over }$ . Define  $V = K_{h_{\mu}}(T_{11} - )$  and W = EV - 1. Simple calculation shows that

$$\mathbf{E}\hat{u}_{0}\mathbf{E}\hat{u}_{2} - (\mathbf{E}\hat{u}_{1})^{2} = W\mathbf{E} V[(T_{11} - ) - W^{-1}\mathbf{E}\{V(T_{11} - )\}]^{2} h_{\mu}^{2}$$

uniformly over all T.

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The following lemma is used to establish Theorems 4.1 and 4.2 under the random or hybrid design. Its proof is similar to that for Lemma S.4.1 and thus is omitted.

**Lemma S.4.2** (covariance, random). Suppose that Assumptions 2.1, 2.2, 3.1, 4.4 and 4.5. Under either of additional Assumptions 4.1 and 4.3, if  $h_c = 0$  and  $nm^2h_c^2$ , we have

$$\begin{split} \sup_{s,t\in\mathcal{T}} \mathbf{E} \left\{ S_{ab}(s,t) \right\} &= O(1), \\ \sup_{s,t\in\mathcal{T}} S_{ab}(s,t) - \mathbf{E} \left\{ S_{ab}(s,t) \right\} &= o_P(1), \\ \inf_{s,t\in\mathcal{T}} \left\{ (S_{20}S_{02} - S_{11}^2)S_{00} - (S_{10}S_{02} - S_{01}S_{11})S_{10} + (S_{10}S_{11} - S_{01}S_{20})S_{01} \right\} &= 1 + o_P(1), \\ \sup_{s,t\in\mathcal{T}} \mathbf{E} \left[ \frac{1}{m(m-1)} \sum_{i\neq k} K_{hc}(s-T_{ij})K_{hc}(t-T_{ik}) \right]^2 &= O \left[ 1 + \frac{1}{m^2h_c^2} \right]. \end{split}$$

The next lemma is used to prove the convergence rates of the mean and covariance estimators under the deterministic design.

**Lemma S.4.3** (mean and covariance, deterministic). Suppose that Assumptions 4.4(c)(d) and 4.2 hold, and *K* is decreasing on [0,1]. If  $nmh_{\mu}$  and  $h h_{\mu}$ , then

- (a)  $\sup_{\in \mathcal{T}} \hat{u}_k() = O(h_{\mu}^k)$  for k = 0, 1, 2;
- (b)  $\sup_{\in \mathcal{T}} \hat{u}_k(\cdot) = h_u^k$  for k = 0, 2;

 $_{375}$  (c) inf  $_{\epsilon T} \stackrel{2}{_0} ($  )  $h_{U}^2$ .

If  $nm^2h_c^2$  , then

- (d)  $\sup_{s,t \in T} S_{ab}(s, t) = O(1)$ ;
- (e)  $\inf_{S,t\in\mathcal{T}}\{(S_{20}S_{02} S_{11}^2)S_{00} (S_{10}S_{02} S_{01}S_{11})S_{10} + (S_{10}S_{11} S_{01}S_{20})S_{01}\} = 1$

(f)  $\sup_{s,t\in\mathcal{T}} i,j\neq k} i K_{hc}(s-T_{ij}) K_{hc}(t-T_{ik}) \frac{T_{ij}-s}{h_c}^a \frac{T_{ik}-t}{h_c}^b$  1,

<sup>380</sup> where  $S_{ab}$  is defined in (S.7).

*Proof.* Part (a) can be verified by simple calculation. For part (b), we fix T and let  $W = {}_{ij} K \frac{I_{ij}-}{h_{\mu}}$ . The assumptions on the kernel function imply that  $K(u) = c_0$  on [-3 4, 3 4] for some constant  $c_0 > 0$  depending only on K. In the sequel, we assume n is su ciently large so that  $nmh_{\mu}$  1. Assumption 4.2 implies that there are at least  $c_1 nmh_{\mu}$  2 points within the interval  $[-3h_{\mu} 4, +3h_{\mu} 4]$ , from which we

deduce that  $W = c_0 c_1 nmh_{\mu} 2 > 0$  regardless of the location of . Let  $w_{ij} = K \frac{T_{ij}}{h_{\mu}} W$ , which is well defined. Observe that

$$\hat{u}_k() = \frac{W}{nmh_{\mu}} W_{ij} (T_{ij} - )^k$$

According the assumptions on the kernel function and Assumption 4.2, at least  $c_1 nmh_{\mu}$  4 of the pairs (i, j) satisfy  $T_{ij} - h_{\mu}$  8 and  $w_{ij} = c_0 W$ . Thus,

$$\hat{u}_{k}(\ ) \quad \frac{W}{nmh_{\mu}}\frac{c_{1}nmh_{\mu}}{4}\frac{c_{0}}{W}\frac{h_{\mu}^{k}}{8^{k}} \quad \frac{c_{0}c_{1}}{2^{3k+2}}h_{\mu}^{k}$$

regardless of the value of . Combining this with the first statement we prove the second statement. The last statement can be established in a similar fashion.

To establish part (c), let  $E = {}_{ij} w_{ij} T_{ij}$ . As  $w_{ij}$  is nonzero if and only if  $T_{ij}$   $( -h_{\mu}, +h_{\mu})$  and  ${}_{ij} w_{ij} = 1$ , we have  $E [ -h_{\mu}, +h_{\mu}]$ . We then observe that

$$\hat{f}_{0}^{2}(\cdot) = \frac{W^{2}}{(nmh_{\mu})^{2}} \qquad W_{ij}(T_{ij} - \cdot)^{2} - \frac{W^{2}}{(nmh_{\mu})^{2}} \qquad W_{ij}(T_{ij} - \cdot)^{2}$$

$$= \frac{W^{2}}{(nmh_{\mu})^{2}} \qquad W_{ij}(T_{ij} - \cdot) - \frac{W^{2}}{\mu_{j'j'}} \qquad W_{ij'}(T_{i'j'} - \cdot)^{2}$$

$$= \frac{W^{2}}{(nmh_{\mu})^{2}} \qquad W_{ij}(T_{ij} - E)^{2}$$

According to Assumption 4.2, there are at least  $c_1 nmh_{\mu}$  4 of  $T_{ij}$  such that  $T_{ij} - E - h_{\mu}$  8 and  $w_{ij} - c_0 W$ . This implies that

$$\hat{c}_{0}^{2}() = \frac{c_{0}c_{1}h_{\mu}^{2}}{256}\frac{W}{nmh_{\mu}} = \frac{c_{0}^{2}c_{1}^{2}}{512}h_{\mu}^{2}$$

regardless of the value of , where the last inequality is due to  $W = c_0 c_1 nmh_{\mu} 2$  that we have deduced previously.

The other statements can be established by similar arguments.

An -cover of a subset *S* of a pseudo-metric space (, d) is a subset *A S* such that for each *p S* there exists a *q A* such that d(p,q). We define  $N(, S, d) = \min\{A \mid A \text{ is an cover of } S\}$  to be the -covering number of *S*, where *A* denotes the cardinality of the set *A*. An -packing of *S* is a subset *A S* such that d(p,q) > for p,q *A*. The -packing number of *S* is defined by  $M(, S, d) = \max\{A \mid A \text{ is an packing of } S\}$ . A standard relation between -covering number and -packing number is M(2, S, d) = N(, S, d) = M(, S, d) for all > 0.

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**Lemma S.4.4.** Let  $(S_1, d_1)$  and  $(S_2, d_2)$  be two pseudo-metric spaces and  $(S_1 \times S_2, d_1 \times d_2)$  the product pseudometric space with the pseudo-metric  $(d_1 \times d_2)(p_1 \times p_2, q_1 \times q_2) = \{a_1^2(p_1, q_1) + d_2^2(p_2, q_2)\}^{1/2}$  for  $p_1 \times p_2, q_1 \times q_2$  $S_1 \times S_2$ . Then  $N(, S_1 \times S_2, d_1 \times d_2) = N(-\overline{2}, S_1, d_1)N(-\overline{2}, S_2, d_2)$ .

Proof of Lemma S.4.4. Let  $A_1$  and  $A_2$  be an  $\overline{2}$ -cover of  $A_1$  and  $A_2$ , respectively. For each k = 1, 2, for every  $p_k$   $S_k$  there exists  $p'_k$   $A_k$  such that  $d_k(p_k, p'_k)$   $\overline{2}$ . Then for each  $p_1 \times p_2$   $S_1 \times S_2$ , we have  $(d_1 \times d_2)(p_1 \times p_2, p'_1 \times p'_2) = \{d_1^2(p_1, p'_1) + d_2^2(p_2, p'_2)\}^{1/2}$ . This shows that  $A = \{p'_1 \times p'_2 \ p'_1 \ A_1, p'_2 \ A_2\}$  is an -cover. The conclusion of the lemma then follows from the observation  $A = N(\overline{2}, S_1, d_1)N(\overline{2}, S_2, d_2)$ .

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and

$$Z_{n}(y, ) = \frac{1}{\overline{n}} \prod_{i=1}^{n} V_{i}(y, ).$$
(S.16)

Then  $\mathbf{E}V_i(y, \cdot) = 0$  and  $U(y, \cdot) - \mathbf{E}U(y, \cdot) = n^{-1/2}Z_n(y, \cdot)$ . Now we observe that

$$V_{i}(y, 1) - V_{i}(z, 2) = \frac{1}{mh_{\mu}} \int_{j=1}^{m} K \frac{T_{ij-1}}{h_{\mu}} - K \frac{T_{ij-2}}{h_{\mu}} d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, T_{ij}) + \frac{1}{mh_{\mu}} \int_{j=1}^{m} K \frac{T_{ij-2}}{h_{\mu}} d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, T_{ij}) - d_{\mathcal{M}}^{2}(Y_{ij}, z) + F^{*}(z, T_{ij}) - \frac{c}{mh_{\mu}} \frac{2^{-1}}{h_{\mu}} + d(y, z) \int_{j=1}^{m} (1_{1} - h_{\mu} \leq T_{ij} \leq 1 + h_{\mu} + 1_{2} - h_{\mu} \leq T_{ij} \leq 2 + h_{\mu}) - c \frac{\max(c_{2}mh_{\mu}, 1)}{mh_{\mu}} d_{h}(y \times 1, z \times 2) - c d_{h}(y \times 1, z \times 2)$$

where  $d_h(y \times 1, z \times 2) = \{h_{\mu}^{-2} - 1^2 + d_{\mathcal{M}}^2(y, z)\}^{1/2}$  defines a distance on the product space  $K \times T$ . With the entropy bound in Lemma S.4.5, by Theorem 3.3 of van de Geer (1990) we deduce that

$$\Pr \sup_{y \in \mathcal{K}, \in \mathcal{B}(t;h)} Z_n(y, ) \quad x \quad \exp(-cx^2),$$
(S.17)

which directly implies that  $\mathsf{E} \sup_{y \in \mathcal{K}, \in B(t;h)} Z_n(y, ) = O(1)$  and further  $\mathsf{E} \sup_{y \in \mathcal{K}, \in B(t;h)} U(y, ) - \mathsf{E}U(y, ) = O(n^{-1/2}).$ 

Next we consider the case  $mh_{\mu}$  0. Let

$$V_{i}(y, ) = \int_{j=1}^{m} \mathcal{K} \frac{T_{ij} - h_{\mu}}{h_{\mu}} d^{2}_{\mathcal{M}}(Y_{ij}, y) - F^{*}(y, T_{ij})$$
(S.18)

and

$$Z_{n}(y, ) = \frac{1}{-nmh_{\mu}} \prod_{i=1}^{n} V_{i}(y, ).$$
(S.19)

Then  $EV_i(y, ) = 0$  and  $U(y, ) - EU(y, ) = (nmh_{\mu})^{-1/2}Z_n(y, )$ . Observe that

$$V_{i}(y, 1) - V_{i}(z, 2) = C \frac{2}{h_{\mu}} + d(y, z) = \prod_{j=1}^{m} (1_{1-h_{\mu} \leq T_{ij} \leq 1+h_{\mu}} + 1_{2-h_{\mu} \leq T_{ij} \leq 2+h_{\mu}})$$
  
$$Cd_{h}(y \times 1, z \times 2),$$

where we use the fact that  $\prod_{j=1}^{m} (1_{1-h_{\mu} \leq T_{ij} \leq 1+h_{\mu}} + 1_{2-h_{\mu} \leq T_{ij} \leq 2+h_{\mu}})$  *c* due to the assumption  $mh_{\mu}$  0 and Assumption 4.2. Note that for all su ciently small  $h_{\mu}$ , there is at most one non-zero item in (S.18) and thus  $Z_n(y, \cdot)$  in (S.19) is sum of independent random variables. In addition, there are only at most  $cnmh_{\mu}$  non-zero terms in (S.19). Based on Theorem 3.3 of van de Geer (1990) again we see that (S.17) holds, which implies that  $E \sup_{y \in \mathcal{K}, \in B(t;h)} Z_n(y, \cdot) = O(1)$  and further  $E \sup_{y \in \mathcal{K}, \in B(t;h)} U(y, \cdot) = O(1) \sqrt{\frac{1}{\sqrt{nmh_{\mu}}}}$ .

To establish (S.13), let  $R = h_{\mu}^{-1} \top = O(h_{\mu}^{-1})$  and  $A_1, \ldots, A_R$  a partition of  $\top$  with  $A_r = h_{\mu}$ . According

to (S.17), we observe that, in either case of  $mh_{\mu}$  1 and  $mh_{\mu}$  0,

$$\Pr \sup_{y \in \mathcal{K}, \ \in \mathcal{T}} Z_n(y, \ ) \quad x \quad \overline{\log n} \quad \Pr_{r=1}^R \Pr \sup_{y \in \mathcal{K}, \ \in A_r} Z_n(y, \ ) \quad x \quad \overline{\log n}$$
$$= O(h_\mu^{-1}) \exp(-cx \log n) \quad O(n^{-1}h_\mu^{-1}) n^{1-x}$$
$$= O(1) n^{1-x},$$

which then implies (S.13).

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The following lemma is used to establish Theorems 4.1 and 4.2 under the deterministic design. Its proof is similar to that of Lemma S.4.6 and thus is omitted.

Lemma S.4.7 (covariance, deterministic). Suppose that Assumptions 2.1, 2.2, 3.1, 4.4, 4.5, 4.6 and 4.2 hold. Let

$$U(s,t) = \underset{i \quad j \neq k}{i} (T_{ij}, T_{ik}) \tilde{C}_{i,jk} - C(s,t) - {}_{s}C(s,t)(T_{ij}-s) - {}_{t}C(s,t)(T_{ik}-t) ,$$

where  $(s', t') = K_{h_c}(s-s')K_{h_c}(t-t')$  for  $s', t' \top$ . If  $h_c = 0$  and  $nm^2h_c^2$ , then for all su cient small  $h_c$ ,

$$\sup_{s,t\in\mathcal{T}} \mathbb{E} \{ U(s,t) - \mathbb{E} U(s,t) \} = O \ n^{-1/2} + (nm^2 h_c^2)^{-1/2} .$$
(S.20)

If  $h_c = 0$ ,  $nh_c^2 = 1$  and  $nm^2h_c^2 = \log n$ , then

$$E \sup_{s \ t \in \mathcal{T}} U(s, t) - EU(s, t) = O \ n^{-1/2} + (nm^2h_c^2)^{-1/2} \ (\log n)^{1/2}.$$
(S.21)

# S.5 Theoretical Results for Regular Design

In a regular design, each sample path is observed on a common set of time points  $\{T_j\}_{1 \le j \le m}$ . From a theoretical perspective, this design is fundamentally di erent from the designs discussed in Section 4, as under such design, m is required for the estimators  $\hat{\mu}$  and  $\hat{C}$  to be consistent. Below we consider both random and deterministic regular design which includes the often-encountered equally-spaced design as a special case.

**Assumption S.5.1** (Regular Random Design). The design points  $\{T_j\}_{1 \le j \le m}$ , independent of other random quantities, are *i.i.d.* sampled from a distribution on T with a probability density that is bounded away from zero and infinity.

Assumption S.5.2 (Regular Deterministic Design). The design points  $\{T_j\}_{1 \le j \le m}$  are nonrandom, and there exist constants  $c_2 = c_1 > 0$ , such that for any intervals A, B = T,

(a)  $c_1 m A - 1$   $\prod_{j=1}^{m} 1_{T_j \in A} \max\{c_2 m A, 1\},\$ 

(b)  $c_1 m^2 A B - 1 = \int_{i,k} 1_{T_i \in A} 1_{T_k \in B} \max\{c_2 m^2 A B, 1\},$ 

where A denotes the length of A.

For any fixed t = T, under either of Assumptions S.5.1 or S.5.2, the number of distinct observed time points in the interval of length  $h_{\mu}$  is  $O(mh_{\mu})$ , and thus the condition of  $mh_{\mu} = 1$  is necessary for consistency

of the mean and covariance estimators. Proposition S.5.1 presents the local and global uniform convergence rates for the mean estimation under regular design. The optimal bandwidth  $h_{\mu} = \frac{1}{m}$  leads to the same convergence rate as that from Cai and Yuan (2011) in the Euclidean case.

**Proposition S.5.1.** Suppose that Assumptions 2.1, 2.2, 3.1, 4.4 and 4.5 hold. Under either of Assumptions S.5.1 or S.5.2, if  $h_{\mu} = 0$  and  $mh_{\mu} > 1 c_1$ , then

$$\sup_{t \in \mathcal{T}} d_{\mathcal{M}}^{2}(\mu(t), \hat{\mu}(t)) = O_{p} \quad h_{\mu}^{4} + \frac{\log n}{n} \quad ,$$

and for any fixed  $t \top$  and  $h = O(h_{\mu})$ ,

$$\sup_{|t| - t| \le h} d_{\mathcal{M}}^{2}(\mu(\ ), \hat{\mu}(\ )) = O_{p} \quad h_{\mu}^{4} + \frac{1}{n} \quad .$$

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In the above, the condition  $mh_{\mu} > 1 c_1$  ensures that there is at least one observation of the time point for the interval  $[t - h_{\mu}, t + h_{\mu}]$  for each *t*, according to Assumption S.5.2 for the regular deterministic design. The proof of Proposition S.5.1 is similar to that of Propositions 4.1 and 4.2, where Lemma S.4.1 is replaced with Lemma S.5.1 below for the regular random design and Lemmas S.4.3 and S.4.6 are replaced with Lemma S.5.2 for the regular deterministic design.

Lemma S.5.1 (mean, regular random). Suppose that Assumptions 2.1, 2.2, 3.1, 4.4, 4.5. Define

$$U = \frac{1}{nm} K_{h_{\mu}}(T_{j} - ) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, ) - F^{*}(y, )(T_{j} - )$$

455 Under Assumption S.5.1, if  $h_{\mu} = 0$  and  $mh_{\mu} = 1$ , then for any fixed  $t = O(h_{\mu})$ ,

- (a)  $\sup_{\in B(t;h)} U \mathbf{E}U = O_p \qquad \overline{\frac{1}{n}}$ ;
- (b)  $\sup_{e \in T} \hat{u}_k() = O_p(h_u^k)$  for k = 0, 1, 2;
- (c) inf  $_{\epsilon B(t;h)} \stackrel{2}{_0} () \quad h_{\mu}^2(1+o_P(1)).$

Proof of Lemma S.5.1. Define the envelop function

$$H = \frac{2\operatorname{diam}(\mathsf{K})^2}{m} \sup_{\substack{j=1 \in B(t;h)}} K_{h_{\mu}}(T_j - ).$$

Since  $mh_{\mu}$  1, simple computation leads to  $E(H^2) = O(1)$  and thus

$$\sup_{e \in B(t;h)} U - \mathbf{E}U = O_p \qquad \frac{1}{n}$$

according to Theorems 2.7.11 and 2.14.2 of van der Vaart and Wellner (1996). Combining above results together, we deduce part (a). Similar technique leads to part (b). For part (c), it is seen that  $\hat{}_0()$  { $E\hat{u}_0E\hat{u}_2 - (E\hat{u}_1)^2$ }(1+ $o_P(1)$ ), where the  $o_P(1)$  component is uniform over . Define  $V = K_{h_{\mu}}(T_{11} - )$  and W = EV - 1. Simple calculation shows that

$$\mathbf{E}\hat{u}_{0}\mathbf{E}\hat{u}_{2} - (\mathbf{E}\hat{u}_{1})^{2} = W\mathbf{E} \ V[(T_{11} - ) - W^{-1}\mathbf{E}\{V(T_{11} - )\}]^{2} \quad h_{u}^{2}$$

uniformly over all T.

Lemma S.5.2 (mean, regular deterministic). Suppose that Assumptions 2.1, 2.2, 3.1, 4.4, 4.5. Define

$$U = \frac{1}{nm} K_{h_{\mu}}(T_{j} - ) d_{\mathcal{M}}^{2}(Y_{ij}, y) - F^{*}(y, ) - F^{*}(y, )(T_{j} - )$$

460 Under Assumption S.5.2, if  $h_{\mu} = 0$  and  $mh_{\mu} > 1$   $c_1$ , then for any fixed  $t = O(h_{\mu})$ ,

- (a)  $\sup_{\in B(t;h)} U \mathbf{E}U = O_p \qquad \frac{1}{n}$ ;
- (b)  $\sup_{e \in T} \hat{u}_k() = O_p(h_u^k)$  for k = 0, 1, 2;
- (c)  $\inf_{eB(t;h)} \hat{}_{0}^{2}() h_{\mu}^{2}$ .

*Proof of Lemma S.5.2.* Part (a) can be established by an argument similar to that of Lemma S.4.6 under the condition  $mh_{\mu}$  1. Part (b) can be verified by simple calculation. For part (c), we fix T and let  $W = {}_{j} K \frac{T_{j}-}{h_{\mu}}$ . The assumptions on the kernel function imply that  $K(u) = c_{0}$  on [-3 4, 3 4] for some constant  $c_{0} > 0$  depending only on K. Assumption S.5.2 implies that there are at least  $3c_{1}mh_{\mu}$  2 points within the interval  $[-3h_{\mu} 4, +3h_{\mu} 4]$ , from which we deduce that  $W = 3c_{0}c_{1}mh_{\mu} 2 > 0$  regardless of the location of . Let  $w_{j} = K \frac{T_{j}-}{h_{\mu}} W$ , which is well defined. Observe that  $\hat{u}_{k}() = \frac{W}{mh_{\mu}} - {}_{j} w_{j} (T_{j} - )^{k}$ . According to the assumptions on the kernel function and Assumption S.5.2, at least  $c_{1}mh_{\mu} 4$  of sampling points  $T_{j}$  satisfy  $T_{j} - h_{\mu} 8$  and  $w_{j} = c_{0} W$ . Thus, for k = 0, 2,

$$\hat{u}_{k}() = \frac{W}{mh_{\mu}} \frac{c_{1}mh_{\mu}}{4} \frac{c_{0}}{W} \frac{h_{\mu}^{k}}{8^{k}} = \frac{c_{0}c_{1}}{2^{3k+2}} h_{\mu}^{k}$$

regardless of the value of . Combining this with part (b) we prove part (c).

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The following theorem provides the pointwise and uniform convergence rates of the covariance estimator under a regular design, where the optimal bandwidth  $h_{\mu}$   $h_{C}$   $\frac{1}{m}$  leads to the same convergence rate as that from Cai and Yuan (2011) in the Euclidean case.

**Theorem S.5.1.** Suppose that Assumptions 2.1, 2.2, 3.1, 4.4, 4.5 and 4.6 hold. Under either of Assumptions 5.5.1 or 5.5.2, if  $h_{\mu} = 0$ ,  $h_{c} = O(h_{\mu})$ ,  $mh_{\mu} > 1 c_{1}$ ,  $m^{2}h_{c}^{2} > 1 c_{1}$  and  $nh_{c} = 1$ , then

$$\sup_{(s,t)\in\mathcal{T}^2} \ \mathscr{P}^{(\mu(s),\mu(t))}_{(\hat{\mu}(s),\hat{\mu}(t))} \hat{\mathbb{C}}(s,t) - \mathbb{C}(s,t) \ \frac{2}{G_{(\mu(s),\mu(t))}} = O_p \ h^4_\mu + h^4_{\mathcal{C}} + \frac{\log n}{n} \ ,$$

and for any fixed s,  $t \top$ ,

$$\mathcal{P}_{(\hat{\mu}(s),\hat{\mu}(t))}^{(\mu(s),\mu(t))}\hat{C}(s,t) - C(s,t) \stackrel{2}{\underset{G(\mu(s),\mu(t))}{=}} = O_p \quad h_{\mu}^4 + h_{\mathcal{C}}^4 + \frac{1}{n}$$

The proof of Theorem S.5.1 is similar to that of Theorems 4.1 and 4.2, where Lemmas S.4.2, S.4.3 and S.4.7 are replaced by Lemma S.5.3 below to analyze the parts  $S_{ab}$  and U. The proof of Lemma S.5.3 is similar to that of Lemmas S.5.1 and S.5.2 and thus omitted.

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Lemma S.5.3 (covariance, regular). Suppose that Assumptions 2.1, 2.2, 3.1, 4.4, 4.5 and 4.6 hold. Define

$$U(s,t) = i (T_{ij}, T_{ik}) \tilde{C}_{i,jk} - C(s,t) - {}_{s}C(s,t)(T_{ij}-s) - {}_{t}C(s,t)(T_{ik}-t) ,$$

where  $(s', t') = K_{h_c}(s-s')K_{h_c}(t-t')$  for  $s', t' \top$ . Under either of Assumptions S.5.1 or S.5.2, if  $h_{\mu} = 0$ ,  $h_c = O(h_{\mu})$ ,  $mh_{\mu} > 1 c_1$ ,  $m^2h_c^2 > 1 c_1$ , and  $nh_c = 1$ , then

- (a)  $\sup_{s,t \in T} \mathbf{E} \{ S_{ab}(s, t) \} = O(1);$
- (b)  $\sup_{s,t\in\mathcal{T}} S_{ab}(s,t) \mathbf{E} \{S_{ab}(s,t)\} = o_P(1);$

(c) 
$$\inf_{s,t\in\mathcal{T}}\{(S_{20}S_{02} - S_{11}^2)S_{00} - (S_{10}S_{02} - S_{01}S_{11})S_{10} + (S_{10}S_{11} - S_{01}S_{20})S_{01}\} + O_P(1)\}$$

(d)  $\sup_{s,t\in\mathcal{T}} U(s,t) - \mathbf{E}U(s,t) = O_p \quad \overline{\frac{1}{n}}$ .

# S.6 Additional Illustration of Invariance

The covariance function and its estimator proposed in our paper are invariant to the manifold parameterization, choice of frame and embedding. This important invariance property is a consequence of the intrinsic perspective we take, and below we demonstrate that it is not shared by non-intrinsic statistical methods.

A method non-invariant to parameterization and frame selection. An "obvious" estimator for C might be obtained by utilizing a frame along  $\hat{\mu}()$  and the coe cient process of Lin and Yao (2019). Specifically, fix a frame along  $\hat{\mu}$  which determines an orthonormal basis of  $T_{\hat{\mu}(t)} M$  for each  $t \in T$ . Then  $\text{Log}_{\hat{\mu}(T_{ij})} Y_{ij}$  can be represented by its coe cient vector  $\hat{c}_{ij}$  with respect to the frame, and  $\hat{C}_{i,jk}$  is also represented by the observed coe cient matrix  $\hat{c}_{ij} \hat{c}_{ik}^{T}$ . Local linear smoothing (Yao et al., 2005) or other

- <sup>485</sup> represented by the observed coe cient matrix  $c_{ij}c_{ik}^{i}$ . Local linear smoothing (Yao et al., 2005) or other smoothing methods can be applied on these matrices to yield an estimated coe cient matrix at any pair (s, t) of time points, and the corresponding estimate  $\hat{C}(s, t)$  is recovered from the estimated coe cient matrix and the frame. However, this estimate is not invariant to the frame, i.e., di erent frames give rise to di erent estimates  $\hat{C}(s, t)$ . As a simple example, consider two frames that coincide on all  $T_{\hat{\mu}(\tau_{ij})}M$  but not on  $T_{\hat{\mu}(s)}M$
- and  $T_{\hat{\mu}(t)}$  M, and assume that *s*, *t* { $T_{ij}$  *i* = 1, ..., *n*, *j* = 1, ..., *m<sub>i</sub>*}. Then the coe cient matrices  $\hat{c}_{ij} \hat{c}_{ik}^{\mathsf{T}}$  with respect of the two frames are identical and thus this "obvious" estimator will produce identical estimated coe cient matrix at the pair (*s*, *t*). However, since the two frames di er at *s* and *t*, the estimates  $\hat{C}(s, t)$ recovered from the estimated coe cient matrix under the two frames are di erent. In addition, smoothing methods optimize certain objective function of the observations which are the frame-dependent coe cient matrices  $\hat{c}_{ij} \hat{c}_{ik}^{\mathsf{T}}$  in this context, while most objective functions, like sum of squared errors, are not invariant

to the frame, and consequently the corresponding estimate is frame-dependent.

We now numerically demonstrate that the above method based on Yao et al. (2005) is not invariant to parameterization and frame selection. For this purpose, we generate data from the two-dimensional sphere  $\mathbb{S}^2 = \{(x, y, z) \mid \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$  with the same setting in Section 5 with sample size n = 100 and sampling rate m = 10. Consider the following three frames:

- The frame  $(B_1(t) = -u, B_2(t) = -v)$  derived from the polar parameterization in Equation (12);
- The frame  $(B_1^4(t), B_2^4(t))$  constructed by

 $B_1^4(t) = \cos(4 t)B_1(t) + \sin(k t)B_2(t), \quad B_2^4(t) = \sin(4 t)B_1(t) + \cos(4 t)B_2(t),$ 

which is a rotated version of  $(B_1(t), B_2(t))$ ;

• The frame  $(\tilde{B}_1(t) = -\frac{1}{t}, \tilde{B}_2(t) = -\frac{1}{t})$  derived from the parameterization in Equation (15).

For each of these frames, we apply the method described above to estimate C under the identical conditions, e.g., with the same logarithmically equidistant grid of bandwidths  $h_c = 0.20, 0.28, 0.40, 0.56, 0.80$  and known

mean function. If the method were invariant to frames, then we would expect to observe *identical* relative root mean integral square error (rRMISE) quantified by

$$rRMISE = \frac{\{E_{\mathcal{T}^2} \ \hat{C}(s,t) - C(s,t) \ ^2_G dsdt\}^{1/2}}{\{ \ _{\mathcal{T}^2} \ C(s,t) \ ^2_G dsdt\}^{1/2}}$$
(S.22)

for any fixed bandwidth. The results, presented in Table S.1 and based on 100 independent Monte Carlo replicates, however, show that di erent frames lead to distinct rRMISE for a fixed bandwidth and distinct minimum rRMISE over a grid of bandwidths, and thus clearly show that the above method based on Yao et al. (2005) is not invariant to frames.

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rRMISE	$h_{\rm C} = 0.20$	$h_{\rm C} = 0.28$	$h_{\rm C} = 0.40$	$h_{\rm C} = 0.56$	$h_{\rm C} = 0.80$
$(B_1(t), B_2(t))$	25.48% (20.20%)	22.71% (19.38%)	20.69% (18.86%)	19.44% (18.29%)	19.94% (17.36%)
$(B_1^{4\pi}(t), B_2^{4\pi}(t))$	116.32% (34.57%)	109.13% (30.08%)	98.24% (23.31%)	91.38% (24.19%)	93.07% (23.65%)
$(\tilde{B}_1(t), \tilde{B}_2(t))$	49.19% (21.77%)	46.63% (20.59%)	43.52% (19.38%)	39.78% (18.04%)	36.67% (16.83%)

Table S.1: rRMISE under di erent frames and bandwidths

A method non-invariant to embedding. We demonstrate that di erent embeddings for the method of Dai et al. (2020) yield distinct estimates of the covariance function. Consider a plane  $M = (0, 1) \times [0, 1]$  with the metric inherited from  $\mathbb{R}^2$ . The underlying population X on M is  $X(t) = (0.25 + 0.5t + Z_1, 0.5 + Z_2)$  where  $Z_1, Z_2$  Uniform(-0.1, 0.1) and  $\mu(t) = (0.25 + 0.5t, 0.5)$ . We generate n = 100 paths from X and for each path we randomly sample Poisson(10) + 2 observations, where Poisson(10) is the Poisson distribution with mean parameter 10. Consider the following three isometric embeddings of M into  $\mathbb{R}^3$ :

1 
$$(x, y)$$
  $(x, y, 0)$ 
 plane;

 2  $(x, y)$   $(\frac{1}{-}\sin(-x), \frac{1}{-}\cos(-x), y)$ 
 half cylindrical surface;

 3  $(x, y)$   $(\frac{1}{2}\sin(2-x), \frac{1}{2}\cos(2-x), y)$ 
 cylindrical surface.

For each of these embeddings, we apply the method of Dai et al. (2020) to produce an estimate of C and calculate rRMISE (S.22) of these estimates, where for illustration, we consider logarithmically equidistant grid of bandwidths  $h_c = 0.10, 0.14, 0.22, 0.33, 0.50$  and the known true mean function  $\mu(t)$ . If the method of Dai et al. (2020) were invariant to embeddings, then we would expect to observe identical rRMISE for these embeddings for each bandwidth. Table S.2 with the rRMISE results based on 100 Monte Carlo simulation replicates, suggesting the opposite, clearly shows that the method of Dai et al. (2020) is not invariant to choices of the frame.

Table S.2: rRMISE under di erent embeddings and bandwidths

rRMISE	<i>h</i> <sub>C</sub> = 0.10	$h_{\rm C} = 0.14$	$h_{\rm C} = 0.22$	$h_{\rm C} = 0.33$	$h_{\rm C} = 0.50$
ιı	26.91% (17.21%)	22.38% (15.72%)	19.50% (14.76%)	17.85% (14.49%)	19.25% (31.40%)
ι2	51.52% (21.16%)	49.22% (19.54%)	47.88% (18.48%)	47.10% (17.92%)	54.57% (90.06%)
ι3	81.83% (29.78%)	79.97% (28.08%)	78.49% (26.78%)	77.15% (25.40%)	90.01% (150.56%)

**A real data example.** We now demonstrate that di erent choices of the frame for the extrinsic method based on Yao et al. (2005) lead to distinct statistical results for the real data analyzed in Section 6. Consider

the frame  $\{(B_k)\}_{1 \le k \le 6}$  induced from the parameterization

and the frame  $\{(\tilde{B}_k)\}_{1 \le k \le 6}$  derived from a rotation of  $\{(B_k)\}_{1 \le k \le 6}$  on each tangent space  $T_{\hat{\mu}(T_{ij})}Sym_{LC}^+$  by

 $\{(\tilde{B}_k)\}_{1 \le k \le 6} = \{(B_k)\}_{1 \le k \le 6} \times \cos(4 - T_{11}) - \sin(4 - T_{12}) - \sin(4 - T_{12})$ 

where diag $(M_1, M_2, M_3)$  for matrices  $M_1, M_2, M_3$  denotes the block diagonal matrix formed by  $M_1, M_2, M_3$ .

For each of these two frames, we apply the extrinsic method based on Yao et al. (2005) to estimate the covariance function and its eigenfunctions under identical conditions, e.g., with the same estimated mean function (with  $h_{\mu}$  = 10) and the same choice of bandwidth  $h_{c}$  = 20. Figure S.4, depicting the first three functional principal components obtained from the two frames, clearly shows that the two frames yield distinct estimates. This demonstrates that the above method based on Yao et al. (2005) is not invariant to frames.

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Figure S.4: The first principal component resulting from the frame  $\{(B_k)\}_{1 \le k \le 6}$  (Row 1) and the frame  $\{(\tilde{B}_k)\}_{1 \le k \le 6}$  (Row 2), the second principal component resulting from the frame  $\{(B_k)\}_{1 \le k \le 6}$  (Row 3) and the frame  $\{(\tilde{B}_k)\}_{1 \le k \le 6}$  (Row 4), and the third principal component resulting from the frame  $\{(B_k)\}_{1 \le k \le 6}$  (Row 5) and the frame  $\{(\tilde{B}_k)\}_{1 \le k \le 6}$  (Row 6). The color encodes fractional anisotropy.

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