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m , 2009

$(X_1, Y_1), \ldots, (X_n, Y_n)$

(1) $$Y = g(X) + \varepsilon,$$

X … $L_2(\mathcal{I})$ … $\mathcal{I} = [0, 1]$, g … $L_2(\mathcal{I})$ … ε … X, … g … g … $g(x) = a + bx$, … a … b … b … $n^{-1/2}$, … (1991), … (2002), … (2003), … (2003), … (2003), … (2005), … (2005), … (2005) … (2007).

… X … g … (1) … (1998) … (2002) … (2003, 2004, 2006) … (2007).

… $L_2(\mathcal{I})$, … $\psi_1, \psi_2, \ldots$, … $V(s, t) = \{X(s), X(t)\}$:

(2) $$V(s, t) = \sum_{j=1} \theta_j \psi_j(u) \psi_j(v),$$

… ψ_j … θ_j … V. … (2) …

$\theta_1 \geq \theta_2 \geq \cdots$. [illegible] ψ_j'[illegible] θ_j'[illegible] $\hat{V}$ [illegible] V,

$$\hat{V}(s,t) = \frac{1}{n}\sum_{i=1}^{n}\{X_i(s) - \bar{X}(s)\}\{X_i(t) - \bar{X}(t)\} = \sum_{j=1}^{\infty}\hat{\theta}_j\hat{\psi}_j(s)\hat{\psi}_j(t), \tag{3}$$

[illegible] $\bar{X} = n^{-1}\sum_i X_i$ [illegible] $\hat{\theta}_1 \geq \hat{\theta}_2 \geq \cdots$. [illegible] $\hat{\theta}_j$ [illegible] $j \geq n+1$, [illegible] $\hat{\psi}_{n+1}, \hat{\psi}_{n+2}, \ldots$ [illegible]

[illegible] X [illegible]

$$X - E(X) = \sum_{j=1}^{\infty} \xi_j \psi_j, \tag{4}$$

[illegible] $\xi_j = \int_{\mathcal{I}} X\psi_j$ [illegible] θ_j. [illegible] $\hat{\psi}_j$ [illegible] ψ_j.

[illegible] 2, [illegible] g [illegible] (1) [illegible] ψ_j. [illegible] 3, [illegible] $\hat{g}$ [illegible] g [illegible] 1 [illegible] 2). [illegible] 3) [illegible] A [illegible] 4, [illegible] 5.

2. P[illegible] [illegible]

$$\hat{g}(x) = \frac{\sum_i Y_i K_i(x)}{\sum_i K_i(x)},$$

[illegible] $K_i(x) = K(\|x - X_i\|/h)$, K [illegible] h [illegible] $\|\cdot\|$ [illegible] L^2 [illegible] (2006) [illegible] g. [illegible] (1), [illegible] X [illegible] g. [illegible] 1 [illegible] 2 [illegible] 4 [illegible] X [illegible] 3 [illegible]

[illegible] (7).

[illegible] ([illegible]).

[illegible] 1. [illegible] K [illegible] $[0, c]$, [illegible] $c > 0$, [illegible] K [illegible] $[0, c]$.

[illegible] g [illegible] x [illegible] g_x [illegible] y [illegible] δ,

$$g(x + \delta y) = g(x) + \delta g_x y + o(\delta)$$

[illegible] $\delta \to 0$. [illegible]

$$g_x = \sum_{j=1}^{\infty} \gamma_{xj} t_j, \tag{5}$$

[illegible] $\gamma_{xj} = g_x \psi_j$ [illegible] t_j [illegible] y [illegible] $y_j = t_j(y) = y\psi_j$. [illegible] γ_{xj} [illegible] g_x [illegible] ψ_j.

[illegible] g_x, [illegible] γ_{xj}, [illegible] $a_x^{m} = (a_{x1}^{m}, a_{x2}^{m}, \ldots)$ [illegible] $a_x^{m} = (a_{x1}^{m}, a_{x2}^{m}, \ldots)$ [illegible] $a = (a_1, a_2, \ldots)$ [illegible] $|g_x a|$, [illegible]

$$g_x a = \sum_{j=1}^{\infty} \gamma$$

[illegible] $\hat{\psi}_j$, [illegible]

(7) $$\hat{\gamma}_{xj} = \frac{\sum\sum^{(j)}_{i_1,i_2} Y_{i_1 i_2} K(i_1, i_2, j|x)}{\sum\sum^{(j)}_{i_1,i_2} \hat{\xi}_{i_1 i_2 j} K(i_1, i_2, j|x)}.$$

[illegible] $\sum\sum^{(j)}_{i_1,i_2}$ [illegible] (i_1, i_2) [illegible] $\hat{\xi}_{i_1 i_2 j} > 0$,

(8) $$K(i_1, i_2, j|x) = K\left(\frac{\|x - X_{i_1}\|}{h_1}\right) K\left(\frac{\|x - X_{i_2}\|}{h_1}\right) K\left(\frac{Q_{i_1 i_2 j}}{h_2}\right),$$

K [illegible] h_1 [illegible] h_2 [illegible] (8), [illegible] (i_1, i_2), [illegible] $X_{i_1} - X_{i_2}$ [illegible] $\hat{\psi}_j$, [illegible] i_1 [illegible] i_2, [illegible] X_{i_1} [illegible] X_{i_2} [illegible] x. [illegible] $\hat{\gamma}_{xj}$ [illegible], h_1 [illegible] h_2.

3. [illegible]

3.1. *Consistency and convergence rates of estimators of* g. [illegible] g [illegible] x [illegible] y [illegible] δ, [illegible]).

[illegible] 2.

(9) $$\sup_{y:\|y\|\le 1} |g(x + \delta y) - g(x)| \to 0 \quad \text{[illegible]} \quad \delta \downarrow 0,$$

[illegible] h [illegible], [illegible] c [illegible] 1,

(10) $$h = h(n) \to 0 \quad \text{[illegible]} \quad nP(\|X - x\| \le c_1 h) \to \infty \quad \text{[illegible]} \quad n \to \infty,$$
$$\text{[illegible]} \ c_1 = c \ \text{[illegible]} \ K(c) > 0, \ \text{[illegible]} \ c_1 \in (0, c).$$

[illegible] $C > 0$, $x \in L_2(\mathcal{I})$ [illegible] $\alpha \in (0, 1]$, [illegible] $\mathcal{G}(C, x, \alpha)$ [illegible] g [illegible] $|g(x + \delta y) - g(x)| \le C\delta^\alpha$, [illegible] $y \in L_2(\mathcal{I})$ [illegible] $\|y\| \le 1$, [illegible] $0 \le \delta \le 1$. [illegible] (9) [illegible] g [illegible] $\mathcal{G}(C, x, \alpha)$.

[illegible] $\mathcal{X} = \{X_1, \ldots, X_n\}$ [illegible].

[illegible] 1. *If Assumptions* 1 *and* 2 *hold, then* $\hat{g}(x) \to g(x)$ *in mean square, conditional on* $\mathcal{X}$, *and*

(11) $$\sup_{g \in \mathcal{G}(C,x,\alpha)} E[\{\hat{g}(x) - g(x)\}^2 | \mathcal{X}] = o_p(1).$$

Furthermore, for all $\eta > 0$,

$$\sup_{g\in\mathcal{G}(C,x,\alpha)} P\{|\hat{g}(x) - g(x)| > \eta\} \to 0.$$

Moreover, if h is chosen to decrease to zero in such a manner that

$$h^{2\alpha} P(\|X - x\| \le c_1 h) \asymp n^{-1} \tag{12}$$

as $n \to \infty$, *then, for each* $C > 0$, *the rate of convergence of* $\hat{g}(x)$ *to* $g(x)$ *equals* $O_p(h^{2\alpha})$, *uniformly in* $g \in \mathcal{G}(C, x, \alpha)$:

$$\sup_{g\in\mathcal{G}(C,x,\alpha)} E[\{\hat{g}(x) - g(x)\}^2|\mathcal{X}] = O_p(h^{2\alpha}), \tag{13}$$

$$\lim_{C_1\to\infty} \limsup_{n\to\infty} \sup_{g\in\mathcal{G}(C,x,\alpha)} P\{|\hat{g}(x) - g(x)| > C_1 h^{\alpha}\} = 0. \tag{14}$$

[illegible] (11) [illegible] (13), [illegible] (X_i, ε_i), [illegible] $1 \le i < \infty$, [illegible] $Y_i = Y_i(g) = g(X_i) + \varepsilon_i$. [illegible] $E_g(\cdot|\mathcal{X})$ [illegible] $(X_i, Y_i(g))$, [illegible] $\mathcal{X}$. [illegible] 5.1 [illegible] (12), [illegible] g [illegible] (2007) [illegible] (14) [illegible].

[illegible] 2. *If the error* ε *in* (1) *is normally distributed, and if, for a constant* $c_1 > 0$, $nP(\|X - x\| \le c_1 h) \to \infty$ *and* (12) *holds, then, for any estimator* $\tilde{g}(x)$ *of* $g(x)$, *and for* $C > 0$ *sufficiently large in the definition of* $\mathcal{G}(C, x, \alpha)$, *there exists a constant* $C_1 > 0$, *such that*

$$\liminf_{n\to\infty} \sup_{g\in\mathcal{G}(C,x,\alpha)} P\{|\tilde{g}(x) - g(x)| > C_1 h^{\alpha}\} > 0.$$

[illegible] $\mathcal{G}(C, x, \alpha)$. [illegible] x [illegible] $P(\|X - x\| \le u)$ [illegible] (12) [illegible] 5.1.

3.2. *Consistency of derivative estimator.* [illegible] $\hat{\gamma}_{xj}$. [illegible]

$$q_{12j} = 1 - \frac{|\int (X_1 - X_2)\psi_j|^2}{\|X_1 - X_2\|^2}$$

[illegible] Q_{12j} [illegible] $\hat{\xi}_{i_1 i_2}$ [illegible] ξ_j [illegible] $\hat{\xi}_{i_1 i_2}$ [illegible] $k_{i_1 i_2 j}$ [illegible] $K(i_1, i_2, j|x)$, [illegible] (8), [illegible] $Q_{i_1 i_2 j}$ [illegible] $q_{i_1 i_2 j}$.

A 3.

( ) $_{t\in\mathcal{I}} E\{X(t)^4\} < \infty$;
( ) m $\theta_1, \ldots, \theta_{j+1}$;
( ) $|g(x+y) - g(x) - g_x y| = o(\|y\|)$ $\|y\| \to 0$;
( ) r $\xi_{1j} - \xi_{2j}$ r r ;
( ) K $[0,1]$, r $0 < K(0) < \infty$;
( ) $h_1, h_2 \to 0$ n $n^{1/2}$ m $(h_1, h_2) \to \infty$ $(nh_1)^2 E(k_{i_1 i_2 j}) \to \infty$.

r X r r V, ψ_j m r $\hat{\psi}_j$ 3.1, r m A m 4( ) $\|\hat{\psi}_j - \psi_j\|$ r r -n m m r m m r .

X r m- $X\psi_j$ (r r r m X) θ_j r r (1) A m 3( ) m $n^{-\varepsilon} = O(h_j)$ r $j = 1, 2$ r $\varepsilon > 0$ r r (2) r r r m n^{-1}. (1) $P(\|x - X\| \le h_1) = O(h_1^{C_1})$ r $C_1 > 0$. A , 3( ) m $nh_1 P(\|x - X\| \le C_2 h_1) \to \infty$ r m $C_2 > 0$, r (1), $nh^{C_1+1} \to \infty$ r $C_1 > 0$. (2) r h_1. r r (1) m $P(q_{12j} \le h_2) = O(h_2^{C_1})$ r $C_1 > 0$, 3( ) m $nP(q_{12j} \le C_2 h_2) \to \infty$ r m $C_2 > 0$, (1), m r .

3. *If Assumption 3 holds, then $\hat{\gamma}_{xj} \to \gamma_{xj}$ in probability.*

(5), $e = \sum_{j=1}^{j_0} e_j \psi_j$ $\sum_j e_j^2 = 1$ $j_0 < \infty$, r e x $g_x e = \sum_j e_j \gamma_{xj}$; (6), e $a_j = e_j, 1 \le j \le j_0, a_j = 0, j > j_0$. A m 3 r $j \le j_0$, m m 3 m r $\hat{g}_x e = \sum_j e_j \hat{\gamma}_{xj}$ x e (., $\hat{g}_x e \to g_x e$ r). A rm r r e, r r $\{\psi_1, \ldots, \psi_{j_0}\}$ m .

r r $\hat{g}_x$, $\hat{g}_x a = \sum_{j \le r} \hat{\gamma}_{xj} a_j$ ($r \ge 1$ r $a = \sum_j a_j \psi_j$), m r r m g_x,

$g_x a = \sum_j \gamma_{xj} a_j$, … A … m … 3 … j, … $\sum_j \gamma_{xj}^2 < \infty$, … (…) … $r = r(n, x)$ … : $r(n, x) \to \infty$ … $n \to \infty$; … $\|a\| < \infty$, $\hat{g}_x a - g_x a \to 0$ … ; … $\hat{g}_x \to g_x$ … $n \to \infty$, … . A … $r(n, x)$, … j … 3, …

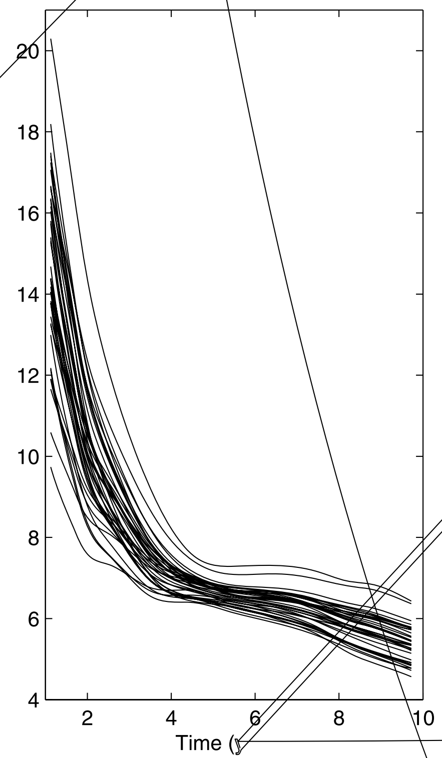
4. A … ca … f f … c … a … de … a … e e … a … g … da a.

… (1958), … (1984), … (1992), … (1998) … (2005) …

… 18 (…), … 10 (…), … r 39 … ? … ? …

… r 39 … (1954) … 10 … 15 … 10 … $\{1, 1.25, 1.5, 1.75, 2, 3, 4, 5, 6, 7, 8, 8.5, 9, 9.5, 10\}$, … $\{s_j\}_{j=1,\ldots,15}$. … $X_{ij} = (h_{i,j+1} - h_{ij})/(t_{j+1} - t_j)$, … h_{ij} … s_j … i …, … $t_j = (s_j + s_{j+1})/2$, $i = 1, \ldots, 39$, $j = 1, \ldots, 14$. … (2003) … (2005).

A … B …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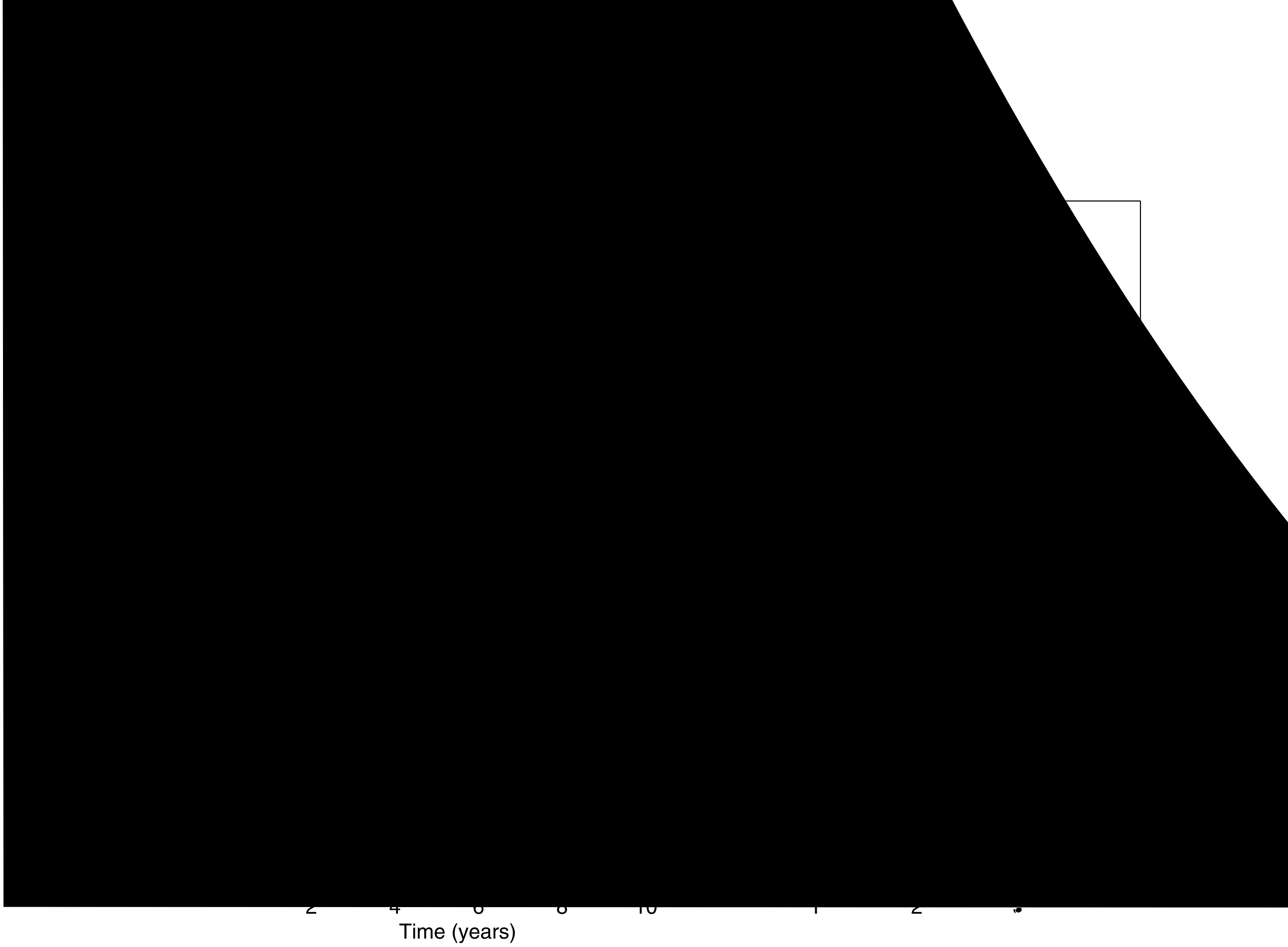


$\overline{\square} Kj$ ψ ($i,$

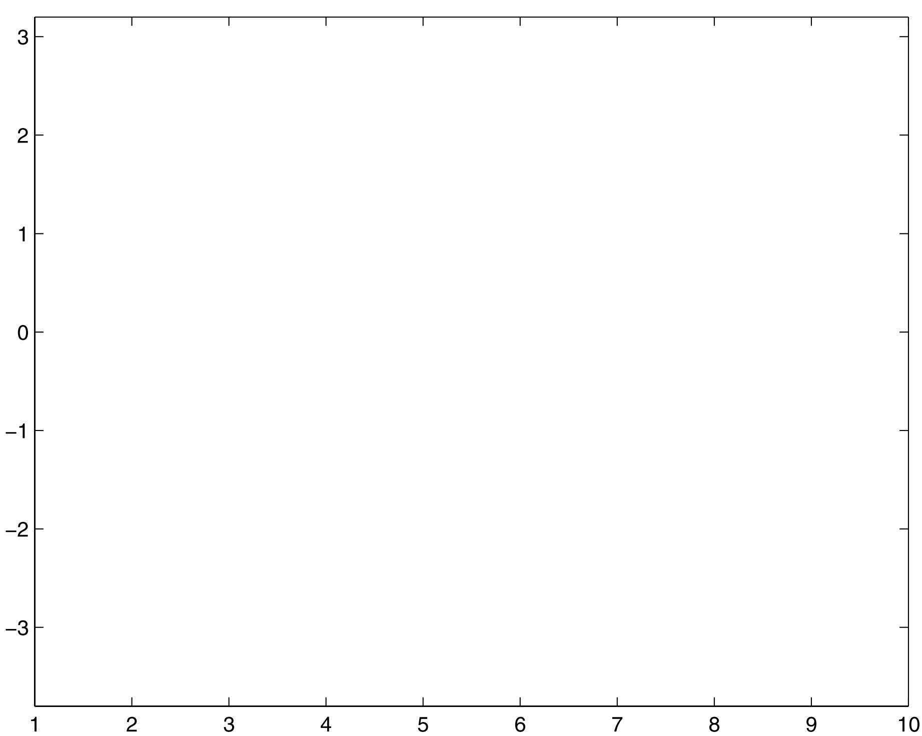
1 .2 Smooth estimates of the first three eigenfunctions for the velocity growth curves.

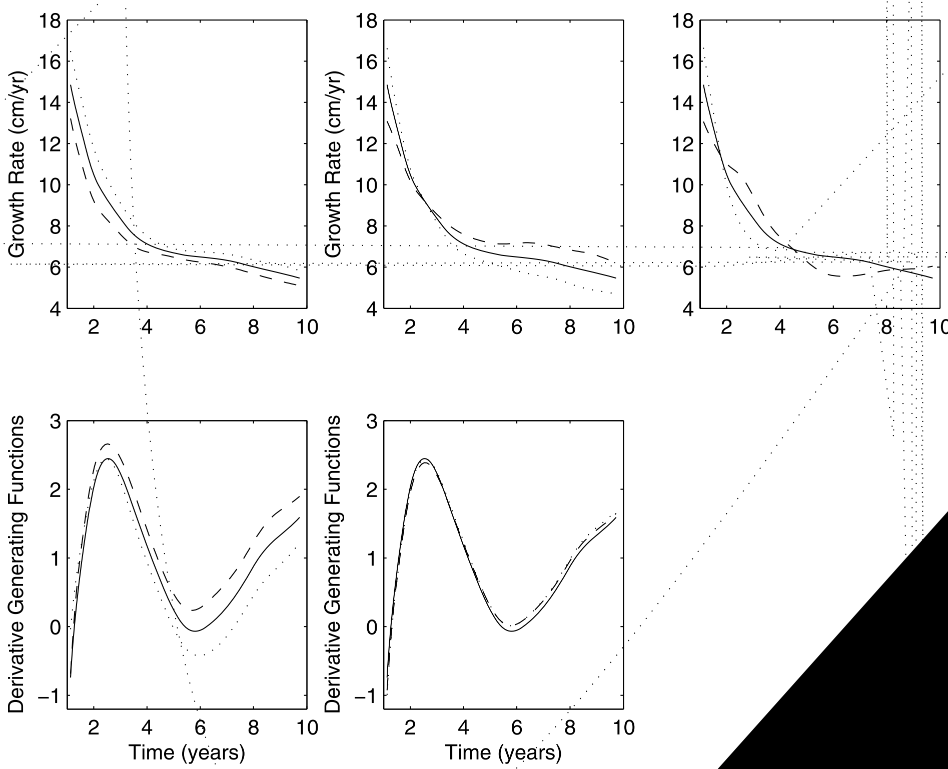
$\int g_i^*(t)z(t)\,dt = \sum_{j=1}^{K} \gamma_{X_i,j} z_j$, ... X_i ... z ... z ... g_i^*. ... g_i^* ... *derivative generating function* ... X_i. ... $K = 3$ ... z ... X_i ... z.

$$\hat{g}_i^*(t) = \sum_{j=1}^{K} \hat{\gamma}_{X_i,j} \hat{\psi}_j(t) \qquad (15)$$

... $\gamma_{X_i,j}$ ... $\psi_j(t)$ ... (3) ... (7).

... $\hat{g}_i^*$ ... $K = 3$ ... 3 ... 39 ... X_i ...




Growth Rate (cm/yr)
Growth Rate (cm/yr)
Derivative Generating Functions
Derivative Generating Functions
Time (years)
Time (years)

[illegible]

5. Add[illegible] a[illegible] e[illegible] [illegible]a[illegible] d[illegible] f[illegible]

5.1. *Bounds on $P(\|X-x\| \le u)$.* [illegible] $P(\|X-x\| \le u)$ [illegible] $u \to 0$ [illegible] u. [illegible] (19) [illegible] $\hat{g}(x)$ [illegible] $g(x)$ [illegible] X [illegible] (14) [illegible] n.

[illegible] X [illegible] X [illegible] $X = \sum_j \xi_j \psi_j$, [illegible] $(\xi_j) = \theta_j$ [illegible] θ_j, $j \ge 1$, [illegible] j [illegible] ξ_j [illegible].

A[illegible] 4. [illegible] $B, \beta > 0$,

$$\theta_j = -Bj^{\beta} + o(j^{\beta}) \quad j \to \infty, \tag{16}$$

[illegible] $\eta_j = \xi_j / \theta_j^{1/2}$ [illegible] η, [illegible]

$$\begin{aligned} &B_1 u^b \le P(|\eta| \le u) \le B_2 u^b \quad [illegible]\ u > 0, \\ &P(|\eta| > u) \le B_3 (1+u)^{-B_4} \quad [illegible]\ u > 0, \quad B_1, \dots, B_4, b > 0. \end{aligned} \tag{17}$$

[illegible] $x = 0$, [illegible] b, B [illegible] β [illegible] (16) [illegible] (17), [illegible]

$$\pi(u) = [illegible] \left\{ -\frac{b\beta}{\beta+1} \left(\frac{2}{B}\right)^{1/\beta} |[illegible]\, u|^{(\beta+1)/\beta} \right\}. \tag{18}$$

[illegible] 4. *If* (16) *and* (17) *hold*, *then*, *with* $\pi(u)$ *given by* (18),

$$P(\|X\| \le u) = \pi(u)^{1+o(1)} \qquad \text{as } u \downarrow 0. \tag{19}$$

[illegible] 1 [illegible] 3, [illegible] θ_j [illegible] (16), [illegible] ξ_j [illegible] (17), [illegible] h [illegible] (12) [illegible]

$\hat{g}(x)$ … $g(x)$ …

$$h^{2\alpha} = \ldots(-2\alpha|\ldots h|)$$
$$= \ldots\left[-\{1+o(1)\}2\alpha\left(\frac{\beta+1}{b\beta}\right)^{\beta/(\beta+1)}\left(\frac{B}{2}\right)^{1/(\beta+1)}(\ldots n)^{\beta/(\beta+1)}\right].$$

… β, … n^{-1}, … β … A … (16) … (17) … $\theta_j = \ldots(-Bj^{\beta})$, … η … 4 … $\{\phi_j\}$ … $\beta = b = 1$ … $\pi(u) = \ldots\{-c(\ldots u)^{(\beta+1)/\beta}\} = u^{-c(\ldots u)^{1/\beta}}$, … $c > 0$, … 0. … η … X …

… 4 … $x = 0$, … $P(\|X - x\| \le u)$ … u, … (19) … $x = 0$ … X … $X_1 - X_2$, … (19) … (19), … $P(\|X_1 - x\| \le u)$ … x … X_2. … (19) … $x = 0$. … x, … $x = \sum_j \theta_j^{1/2} x_j$ … x_j’ … x_j’ …

… θ_j … (16), … $P(\|X - x\| \le u)$ … $\ldots(-C_1 u^{-C_2})$ … u … 0, … $\ldots(-C_1|\ldots u|^{C_2})$ … 3 … $C_1, C_2 > 0$. … $x = 0$, … (2003), … (1952). … $u^{C_3}\ldots(-C_1 u^{-C_2})$ … $C_3 > 0$ … (2007), …

…

5.2. *Proof of Theorem* 1. … σ^2 … ε … (1). … $N_j = \sum_i K_i(x)^j$ … $j = 1, 2,$ … $N_2 \le K(0)N_1$, … $K(\cdot)$ …

m [0, c].

$$E[\{\hat{g}(x)-g(x)\}^2|\mathcal{X}]$$

$$= [E\{\hat{g}(x)|\mathcal{X}\} - g(x)]^2 + \mathrm{var}(\hat{g}(x)|\mathcal{X}) \tag{20}$$

$$\le \max_{i=1,\ldots,n} |g(X_i)-g(x)|I(\|X_i-x\|\le ch) + \frac{\sigma^2 \sum_i K_i^2(x)}{\{\sum_i K_i(x)\}^2}$$

$$\le \sup_{y:\|y\|\le ch} |g(x)-g(x+y)|^2 + \frac{\sigma^2 K(0)}{N_1}.$$

g x (9) (20) $K_i(x) \ge K_i(x)I(\|X_i - x\| \le c_1 h) \ge K(c_1)I(\|X_i - x\| \le c_1 h)$, c_1 (A2). (10) $N_1^{-1} \to 0$ 1, $E(N_i^{-1}) \to 0$. (20), (13) (14), (20)

$$\sup_{g\in\mathcal{G}(C,x,\alpha)} E[\{\hat{g}(x)-g(x)\}^2|\mathcal{X}] \le C^2(ch)^{2\alpha} + \frac{\sigma^2 K(0)}{N_1}$$

$$\le C^2(ch)^{2\alpha} + \frac{\sigma^2 K(0)\{1+o_p(1)\}}{K(c_1)nP(\|X-x\|\le c_1 h)}$$

$E(N_1^{-1}) \le E[\{\sum_i I(\|X_i - x\| \le c_1 h)\}^{-1}] \asymp \{nP(\|X-x\|\le c_1 h)\}^{-1}$.

5.3. *Proof of Theorem* 2. $x = 0$. f B_1, $[-B_2, B_2]$, f B_3. $g_1 \equiv 0$ $g_2(y) = h^\alpha f(\|y\|/h)$. $\|y\| \le h$ $0 < \alpha \le 1$,

$$|g_2(y) - g_2(0)| = h^\alpha |f(\|y\|/h) - f(0)| \le h^\alpha B_1\|y\|/h \le h^\alpha B_1(\|y\|/h)^\alpha$$

$$= B_1\|y\|^\alpha,$$

$\|y\| > h$,

$$|g_2(y) - g_2(0)| \le 2h^\alpha B_3 \le 2B_3\|y\|^\alpha.$$

$g_2 \in \mathcal{G}(C, 0, \alpha)$ $\max(B_1, 2B_3) \le C$.

(21) $P(\rho > 1)$ … 1 … $n \to \infty$.

$$2\rho = \sum_{i=1}^{n} \{g_2(X_i)^2 - 2\varepsilon_i g_2(X_i)\},$$

… $\mathcal{X}$, … $s_n^2 = \sum_i g_2(X_i)^2$ … $4s_n^2$. … (21) …

$$\lim_{B\to\infty} \lim_{n\to\infty} P(s_n^2 > B) = 0 \tag{22}$$

… 2 … (22).

… B_2 … f … $0 < B \le c_1$, … $|g_2(x)| \le B_3 h^{\alpha} I(\|x\| \le c_1 h)$, …

$$s_n^2 \le B_3^2 h^{2\alpha} \sum_{i=1}^{n} I(\|X_i\| \le c_1 h). \tag{23}$$

… $nP(\|X\| \le c_1 h) \to \infty$, …

$$\frac{\sum_i I(\|X_i\| \le c_1 h)}{nP(\|X\| \le c_1 h)} \to 1$$

… (12) … (23) … (22).

5.4. *Proof of Theorem* 3. … $K_{i_1 i_2 j}$ … $K(i_1, i_2, j|x)$. … 3(…) …

$$K_{i_1 i_2 j} = 0, \quad \text{…: } \|X_{i_1} - x\| \le h_1, \quad \|X_{i_2} - x\| \le h_1 \ \text{…}\ Q_{i_1 i_2} \le h_2. \tag{24}$$

… $\delta > 0$, … $s(\delta)$ … $|g(x+y) - g(x) - g_x y|$ … y … $\|y\| \le \delta$. … 3(…),

$$\delta^{-1} s(\delta) \to 0 \quad \text{…} \quad \delta \downarrow 0. \tag{25}$$

… $\mathcal{E}_{i_1 i_2}$ … $\|X_{i_k} - x\| \le h_1$ … $k = 1, 2$. … $\mathcal{E}_{i_1 i_2}$ …,

$$|g(X_{i_1}) - g(X_{i_2}) - g_x(X_{i_1} - X_{i_2})| \le 2s(h_1).$$

… $\varepsilon_{i_1 i_2} = \varepsilon_{i_1} - \varepsilon_{i_2}$ … $\mathcal{E}_{i_1 i_2}$,

$$|Y_{i_1} - Y_{i_2} - \{g_x(X_{i_1} - X_{i_2}) + \varepsilon_{i_1 i_2}\}| \le 2s(h_1).$$

[illegible] $\xi_{i_1i_2j}=\xi_{i_1j}-\xi_{i_2j}$, [illegible] $g_x(X_{i_1}-X_{i_2})=\sum_k \xi_{i_1i_2k}\gamma_{xk}$, [illegible] (24), [illegible],

$$\left|\sum_{i_1,i_2}^{(j)}(Y_{i_1}-Y_{i_2})K_{i_1i_2j} - \left\{\sum_{i_1,i_2}^{(j)}K_{i_1i_2j}\sum_{k=1}^{\infty}\xi_{i_1i_2k}\gamma_{xk}+\sum_{i_1,i_2}^{(j)}\varepsilon_{i_1i_2}K_{i_1i_2j}\right\}\right| \le 2s(h_1)\sum_{i_1,i_2}^{(j)}K_{i_1i_2j}. \tag{26}$$

[illegible],

$$|\hat{\xi}_{i_1i_2j}-\xi_{i_1i_2j}| = \left|\int (X_{i_1}-X_{i_2})(\hat{\psi}_j-\psi_j)\right| \le \|X_{i_1}-X_{i_2}\|\|\hat{\psi}_j-\psi_j\| \le 2h_1\|\hat{\psi}_j-\psi_j\|, \tag{27}$$

[illegible] $\varepsilon_{i_1i_2}$ [illegible] (24), (26) [illegible] (27), [illegible]

$$\left|\sum_{i_1,i_2}^{(j)}(Y_{i_1}-Y_{i_2})K_{i_1i_2j} - \left\{\gamma_{xj}\sum_{i_1,i_2}^{(j)}\hat{\xi}_{i_1i_2j}K_{i_1i_2j} + \sum_{i_1,i_2}^{(j)}K_{i_1i_2j}\sum_{k:k\neq j}\xi_{i_1i_2k}\gamma_{xk}+\sum_{i_1,i_2}^{(j)}\varepsilon_{i_1i_2}K_{i_1i_2j}\right\}\right| \le 2\{s(h_1)+|\gamma_{xj}|h_1\|\hat{\psi}_j-\psi_j\|\}\sum_{i_1,i_2}^{(j)}K_{i_1i_2j}. \tag{28}$$

[illegible],

$$\left|\sum_{i_1,i_2}^{(j)}K_{i_1i_2j}\sum_{k:k\neq j}\xi_{i_1i_2k}\gamma_{xk}\right| = \left|\sum_{i_1,i_2}^{(j)}K_{i_1i_2j}\sum_{k:k\neq j}\gamma_{xk}\int_{\delta}(X_{i_1}-X_{i_2})\psi_k\right| \le \sum_{i_1,i_2}^{(j)}K_{i_1i}\ 34\ 0\ 2\ .$$

$$\leq \|g_x\| \sum_{i_1,i_2} \square^{(j)} K$$

(32) … $\hat{\gamma}_{xj}$ … (7). … (27) …

$$\sum^{(j)}_{i_1,i_2} \hat{\xi}_{i_1 i_2 j} K_{i_1 i_2 j} \ge \sum^{(j)}_{i_1,i_2} \max(0, \xi_{i_1 j} - \xi_{i_2 j} - 2h_1 \|\hat{\psi}_j - \psi_j\|) K_{i_1 i_2 j}$$

$$\ge \sum^{(j)}_{i_1,i_2} \max(0, \xi_{i_1 j} - \xi_{i_2 j}) K_{i_1 i_2 j} - 2h_1 \|\hat{\psi}_j - \psi_j\| \sum^{(j)}_{i_1,i_2} K_{i_1 i_2 j}. \tag{33}$$

… $\sum\sum^{(j)}_{i_1,i_2}$ … (i_1, i_2) … $\hat{\xi}_{i_1 i_2 j} > 0$. … 4( ), ( ) … ( ), … $B > 0$,

$$\sum^{(j)}_{i_1,i_2} \max(0, \xi_{i_1 j} - \xi_{i_2 j}) K_{i_1 i_2 j} \ge \{1 + o_p(1)\} B h_1 \sum^{(j)}_{i_1,i_2} K_{i_1 i_2 j}. \tag{34}$$

… 3( ), $n^{-1/2}/\min(h_1, h_2) \to 0$. … 3( ) … ( ), …

$$\|\hat{\psi}_j - \psi_j\| = O_p(n^{-1/2}). \tag{35}$$

… (33) … (35) …

$$\sum^{(j)}_{i_1,i_2} \hat{\xi}_{i_1 i_2 j} K_{i_1 i_2 j} \ge \{1 + o_p(1)\} B h_1 \sum^{(j)}_{i_1,i_2} K_{i_1 i_2 j} \tag{36}$$

… B … (34). … (7). … (7), (25), (32) … (36), …

$$\hat{\gamma}_{xj} = \gamma_{xj} + O_p\left(\frac{\sum\sum^{(j)}_{i_1,i_2} \varepsilon_{i_1 i_2} K_{i_1 i_2 j}}{h_1 \sum\sum^{(j)}_{i_1,i_2} K_{i_1 i_2 j}}\right) + o_p(1). \tag{37}$$

… (37), … X_i …

$$O_p\left[\left\{h_1^2 \sum^{(j)}_{i_1,i_2} K_{i_1 i_2 j}\right\}^{-1}\right] = O_p[\{(nh_1)^2 E(k_{i_1 i_2 j})\}^{-1}] = o_p(1),$$

… 3( ). … (37) … $\hat{\gamma}_{xj} = \gamma_{xj} + o_p(1)$, … 3.

5.5. *Proof of Theorem* 4. $t \in (0,1)$ $D_t = (\sum_j \theta_j^{1-t})^{-1}$,

$$(38)\qquad P(\|X\| \le u) = P\left(\sum_{j=1}^{\infty} \theta_j \eta_j^2 \le u^2\right) \begin{cases} \le \prod_{j=1}^{\infty} P(\theta_j \eta_j^2 \le u^2), \\ \ge \prod_{j=1}^{\infty} P(\theta_j^t \eta_j^2 \le D_t u^2), \end{cases}$$

$$P\left(\sum_{j=1}^{\infty} \theta_j \eta_j^2 \le u^2\right) = P\left(\sum_{j=1}^{\infty} \theta_j^{1-t}(\theta_j^t \eta_j^2 - D_t u^2) \le 0\right)$$
$$\ge P(\theta_j^t \eta_j^2 \le D_t u^2 \ \ j).$$

$J = J(u)$ $u/\theta_j^{1/2} \le \zeta$, ζ $B_1 u^b \le P(|\eta| \le u) \le B_2 u^b$ $0 \le u \le \zeta$.

$$\prod_{j=1}^{\infty} P(\theta_j \eta_j^2 \le u^2) \le \prod_{j=1}^{J} P(|\eta| \le u\theta_j^{-1/2})$$
$$= u^{bJ} \left(\frac{1}{2} bB \sum_{j=1}^{J} j^{\beta} + o(J^{\beta+1})\right)$$
$$(39)\qquad = \left(-\frac{bB\beta}{2(\beta+1)} J^{\beta+1} + o(J^{\beta+1})\right)$$
$$= \pi(u)^{1+o(1)}$$

$u \downarrow 0$, π (18).

J $D_t^{1/2} u/\theta_j^{t/2} \le \zeta$. (39),

$$\prod_{j=1}^{J} P(\theta_j^t \eta_j^2 \le D_t u^2)$$
$$(40)\qquad = \left(-\frac{b\beta}{\beta+1}\left(\frac{2}{Bt}\right)^{1/\beta} |\ \ u|^{(\beta+1)/\beta} + o\left(|\ \ u|^{(\beta+1)/\beta}\right)\right)$$
$$= \pi(u)^{t^{-1/\beta}+o(1)}.$$

A , $j \ge J+1$,

π

[illegible] $B_5 = B_5(t) \in (0,1)$, [illegible] $\pi_j \in (0, B_5)$ [illegible] $j \geq J+1$,

$$1 - \pi_j = \exp\left(-\sum_{k=1}^{\infty} \frac{\pi_j^k}{k}\right) \geq \exp(-B_6 \pi_j)$$

[illegible]

$$\prod_{j=J+1}^{\infty} (1-\pi_j) \geq \exp\left(-B_6 \sum_{j=J+1}^{\infty} \pi_j\right) \geq \exp\left(-B_7 \sum_{j=J+1}^{\infty} (\theta_j^{t/2}/u)^{B_4}\right),$$

[illegible] (40). [illegible] (40), [illegible] $t \in (0,1)$ [illegible] (40), [illegible] 1, [illegible] $u \downarrow 0$,

$$(42) \qquad \prod_{j=1}^{\infty} P(\theta_j^t \eta_j^2 \leq D_t u^2) = \pi(u)^{1+o(1)}.$$

[illegible] (38), (39) [illegible] (42) [illegible] (19).

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