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, 2009

ESTIMATION OF FUNCTIONA.3041.168T

$(X_1, Y_1), \ldots, (X_n, Y_n),$

$$Y = g(X) + \varepsilon, \tag{1}$$

X $L_2(\mathcal{I})$ $\mathcal{I} = [0, 1]$, g $L_2(\mathcal{I})$ ε X, g g $g(x) = a + bx$, a b $-b$ $n^{-1/2}$, (1991), (2002), (2003), (2003), (2005), (2005), (2005) (2007).

X g (1) (1998) (2002) (2003, 2004, 2006) (2007).

$L_2(\mathcal{I})$, $\psi_1, \psi_2, \ldots,$, $V(s, t) = \{X(s), X(t)\}$:

$$V(s, t) = \sum_{j=1}^{\infty} \theta_j \psi_j(u) \psi_j(v), \tag{2}$$

ψ_j θ_j V. (2)

$\theta_1 \geq \theta_2 \geq \cdots$. [illegible] ψ_j [illegible] θ_j [illegible] V, [illegible]

$$\hat{V}(s,t) = \frac{1}{n} \sum_{i=1}^{n} \{X_i(s) - \bar{X}(s)\}\{X_i(t) - \bar{X}(t)\} = \sum_{j=1}^{\infty} \hat{\theta}_j \hat{\psi}_j(s) \hat{\psi}_j(t), \tag{3}$$

[illegible] $\bar{X} = n^{-1} \sum_i X_i$ [illegible] $\hat{\theta}_1 \geq \hat{\theta}_2 \geq \cdots$. [illegible] $\hat{\theta}_j$ [illegible] $j \geq n+1$, [illegible] $\hat{\psi}_{n+1}, \hat{\psi}_{n+2}, \ldots$ [illegible] X [illegible]

$$X - E(X) = \sum_{j=1}^{\infty} \xi_j \psi_j, \tag{4}$$

[illegible] $\xi_j = \int_{\mathcal{I}} X \psi_j$ [illegible] θ_j. [illegible] $\hat{\psi}_j$ [illegible] ψ_j.

[illegible] 2, [illegible] g [illegible] (1) [illegible] ψ_j. [illegible] 3, [illegible] $\hat{g}$ [illegible] g [illegible] (1 2). [illegible] (3) [illegible]. A [illegible] B [illegible] 4, [illegible] 5.

2. P[illegible] ed e t [illegible] at [illegible] ced [illegible] e.

[illegible]

$$\hat{g}(x) = \frac{\sum_i Y_i K_i(x)}{\sum_i K_i(x)},$$

[illegible] $K_i(x) = K(\|x - X_i\|/h)$, K [illegible] h [illegible] $\|\cdot\|$ [illegible] L^2 [illegible] (2006) [illegible] g. [illegible] (1), [illegible] X [illegible] g. [illegible] 1 2 [illegible] 4 [illegible] X [illegible] 3 [illegible]

[illegible] (7).

[illegible] ([illegible]).

A [illegible] 1. [illegible] K [illegible] $[0, c]$, [illegible] $c > 0$, [illegible] K [illegible] $[0, c]$.

[illegible] g [illegible] x [illegible] g_x [illegible] y [illegible] δ,

$$g(x + \delta y) = g(x) + \delta g_x y + o(\delta)$$

$\delta \to 0$. [illegible]

$$g_x = \sum_{j=1}^{\infty} \gamma_{xj} t_j, \tag{5}$$

[illegible] $\gamma_{xj} = g_x \psi_j$ [illegible] t_j [illegible] y [illegible] $y_j = t_j(y) = y\psi_j$. [illegible] γ_{xj} [illegible] g_x [illegible] ψ_j. [illegible] g_x, [illegible] γ_{xj}, [illegible] $a_x = (a_{x1}, a_{x2}, \ldots)$ [illegible] $a_x = (a_{x1}, a_{x2}, \ldots)$ [illegible] $a = (a_1, a_2, \ldots)$ [illegible] $|g_x a|$, [illegible]

$$g_x a = \sum_{j=1}^{\infty} \gamma$$

$\hat{\psi}_j$,

$$\hat{\gamma}_{xj} = \frac{\sum_{i_1,i_2}^{(j)} Y_{i_1 i_2} K(i_1, i_2, j|x)}{\sum_{i_1,i_2}^{(j)} \hat{\xi}_{i_1 i_2 j} K(i_1, i_2, j|x)}. \tag{7}$$

$\sum_{i_1,i_2}^{(j)}$ (i_1, i_2) $\hat{\xi}_{i_1 i_2 j} > 0$,

$$K(i_1, i_2, j|x) = K\left(\frac{\|x - X_{i_1}\|}{h_1}\right) K\left(\frac{\|x - X_{i_2}\|}{h_1}\right) K\left(\frac{Q_{i_1 i_2 j}}{h_2}\right), \tag{8}$$

K h_1 h_2 (8), (i_1, i_2), $X_{i_1} - X_{i_2}$ $\hat{\psi}_j$, i_1 i_2, X_{i_1} X_{i_2} x. $\hat{\gamma}_{xj}$ h_1 h_2.

3. The theoretical results.

3.1. *Consistency and convergence rates of estimators of g.* g x (y δ).

A 2.

$$\sup_{y:\|y\|\le 1} |g(x + \delta y) - g(x)| \to 0 \quad \delta \downarrow 0, \tag{9}$$

h c A 1,

$$\begin{aligned} &h = h(n) \to 0 \quad nP(\|X - x\| \le c_1 h) \to \infty \quad n \to \infty, \\ &c_1 = c \quad K(c) > 0, \quad c_1 \in (0, c). \end{aligned} \tag{10}$$

$C > 0$, $x \in L_2(\mathcal{I})$ $\alpha \in (0, 1]$, $\mathcal{G}(C, x, \alpha)$ g $|g(x + \delta y) - g(x)| \le C\delta^\alpha$, $y \in L_2(\mathcal{I})$ $\|y\| \le 1$, $0 \le \delta \le 1$. (9) g $\mathcal{G}(C, x, \alpha)$.

$\mathcal{X} = \{X_1, \dots, X_n\}$.

1. *If Assumptions 1 and 2 hold, then $\hat{g}(x) \to g(x)$ in mean square, conditional on $\mathcal{X}$, and*

$$\sup_{g \in \mathcal{G}(C,x,\alpha)} E[\{\hat{g}(x) - g(x)\}^2 | \mathcal{X}] = o_p(1). \tag{11}$$

Furthermore, for all $\eta > 0$,

$$\sup_{g \in \mathcal{G}(C,x,\alpha)} P\{|\hat{g}(x) - g(x)| > \eta\} \to 0.$$

Moreover, if h *is chosen to decrease to zero in such a manner that*

$$h^{2\alpha} P(\|X - x\| \le c_1 h) \asymp n^{-1} \tag{12}$$

as $n \to \infty$, *then, for each* $C > 0$, *the rate of convergence of* $\hat{g}(x)$ *to* $g(x)$ *equals* $O_p(h^{2\alpha})$, *uniformly in* $g \in \mathcal{G}(C, x, \alpha)$:

$$\sup_{g \in \mathcal{G}(C,x,\alpha)} E[\{\hat{g}(x) - g(x)\}^2 | \mathcal{X}] = O_p(h^{2\alpha}), \tag{13}$$

$$\lim_{C_1 \to \infty} \lim_{n \to \infty} \sup_{g \in \mathcal{G}(C,x,\alpha)} P\{|\hat{g}(x) - g(x)| > C_1 h^{\alpha}\} = 0. \tag{14}$$

[illegible] (11) [illegible] (13), [illegible] (X_i, ε_i), $1 \le i < \infty$, [illegible] $Y_i = Y_i(g) = g(X_i) + \varepsilon_i$. [illegible] $E_g(\cdot|\mathcal{X})$ [illegible] $(X_i, Y_i(g))$, [illegible] $\mathcal{X}$. [illegible] 5.1 [illegible] (12), [illegible] – A [illegible] g [illegible] (2007) [illegible] (14) [illegible].

[illegible] 2. *If the error* ε *in* (1) *is normally distributed, and if, for a constant* $c_1 > 0$, $nP(\|X - x\| \le c_1 h) \to \infty$ *and* (12) *holds, then, for any estimator* $\tilde{g}(x)$ *of* $g(x)$, *and for* $C > 0$ *sufficiently large in the definition of* $\mathcal{G}(C, x, \alpha)$, *there exists a constant* $C_1 > 0$, *such that*

$$\lim_{n \to \infty} \sup_{g \in \mathcal{G}(C,x,\alpha)} P\{|\tilde{g}(x) - g(x)| > C_1 h^{\alpha}\} > 0.$$

A [illegible] $\mathcal{G}(C, x, \alpha)$. [illegible] x [illegible] $P(\|X - x\| \le u)$ [illegible] (12) [illegible] 5.1.

3.2. *Consistency of derivative estimator.* [illegible] $\hat{\gamma}_{xj}$. [illegible]

$$q_{12j} = 1 - \frac{|\square(X_1 - X_2)\psi_j|^2}{\|X_1 - X_2\|^2}$$

[illegible] Q_{12j} [illegible] $\hat{\xi}_{i_1 i_2}$ [illegible] ξ_j [illegible] $\hat{\xi}_{i_1 i_2}$ [illegible] $k_{i_1 i_2 j}$ [illegible] $K(i_1, i_2, j|x)$, [illegible] (8), [illegible] $Q_{i_1 i_2 j}$ [illegible] $q_{i_1 i_2 j}$.

A 3.

( ) $\sup_{t\in\mathcal{I}} E\{X(t)^4\} < \infty$;
( ) $\theta_1, \ldots, \theta_{j+1}$;
( ) $|g(x+y) - g(x) - g_x y| = o(\|y\|)$ $\|y\| \to 0$;
( ) $\xi_{1j} - \xi_{2j}$;
( ) K $[0,1]$, $0 < K(0) < \infty$;
( ) $h_1, h_2 \to 0$ n $n^{1/2}$ $(h_1, h_2) \to \infty$ $(nh_1)^2 E(k_{i_1 i_2 j}) \to \infty$.

X V, ψ_j $\hat{\psi}_j$ 3.1, ; A 4( ), $\|\hat{\psi}_j - \psi_j\|$ -n . X -$X\psi_j$ (X) θ_j (1), A 3( ) $n^{-\varepsilon} = O(h_j)$ $j = 1, 2$ $\varepsilon > 0$ (2) . , - n^{-1}. , (1) $P(\|x - X\| \le h_1) = O(h_1^{C_1})$ $C_1 > 0$. A , 3( ) $nh_1 P(\|x - X\| \le C_2 h_1) \to \infty$ $C_2 > 0$, , , (1), $nh^{C_1+1} \to \infty$ $C_1 > 0$. (2) h_1. (1) $P(q_{12j} \le h_2) = O(h_2^{C_1})$ $C_1 > 0$, 3( ) $nP(q_{12j} \le C_2 h_2) \to \infty$ $C_2 > 0$, , , (1), .

3. *If Assumption 3 holds, then $\hat{\gamma}_{xj} \to \gamma_{xj}$ in probability.*

(5), $e = \sum_{j=1}^{j_0} e_j \psi_j$ $\sum_j e_j^2 = 1$ $j_0 < \infty$, - e x $g_x e = \sum_j e_j \gamma_{xj}$; (6), e $a_j = e_j$, $1 \le j \le j_0$, $a_j = 0$, $j > j_0$. A - 3 $j \le j_0$, 3 $\hat{g}_x e = \sum_j e_j \hat{\gamma}_{xj}$ x e (., $\hat{g}_x e \to g_x e$). A - e, $\{\psi_1, \ldots, \psi_{j_0}\}$.

$\hat{g}_x$, $\hat{g}_x a = \sum_{j \le r} \hat{\gamma}_{xj} a_j$ ($r \ge 1$ $a = \sum_j a_j \psi_j$), g_x,

[illegible] $g_x a = \sum_j \gamma_{xj} a_j$, [illegible] 3 [illegible] j, [illegible] $\sum_j \gamma_{xj}^2 < \infty$, [illegible] $r = r(n, x)$ [illegible] $: r(n, x) \to \infty$ $n \to \infty$; [illegible] $\|a\| < \infty$, $\hat{g}_x a - g_x a \to 0$ [illegible]; [illegible], $\hat{g}_x \to g_x$ [illegible] $n \to \infty$, [illegible]. A [illegible] $r(n, x)$, [illegible] j [illegible] 3, [illegible].

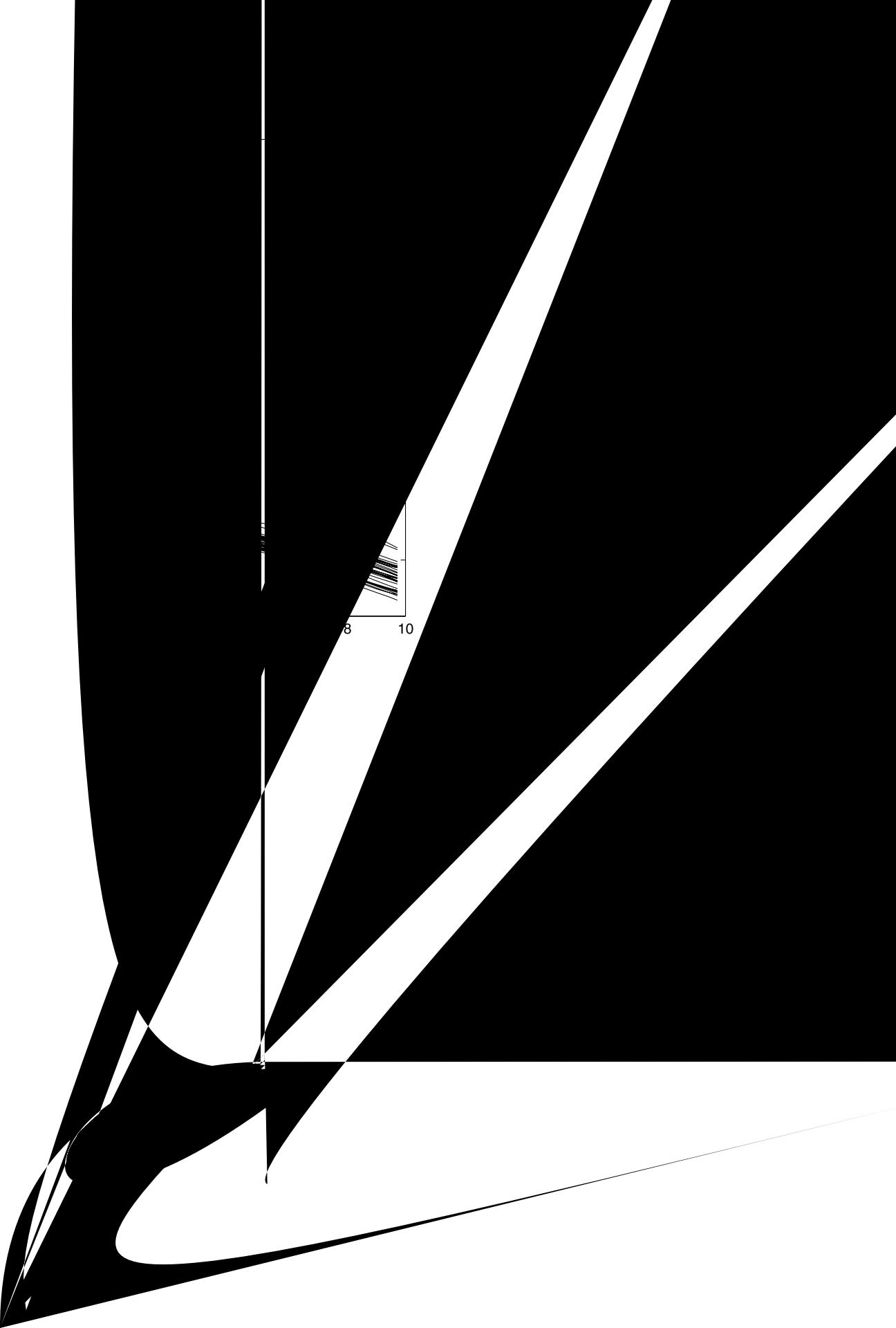
4. A[illegible] cat[illegible] f[illegible]ct[illegible]a de[illegible] at[illegible] e[illegible]t [illegible]at[illegible]t [illegible]g[illegible] t[illegible] data.

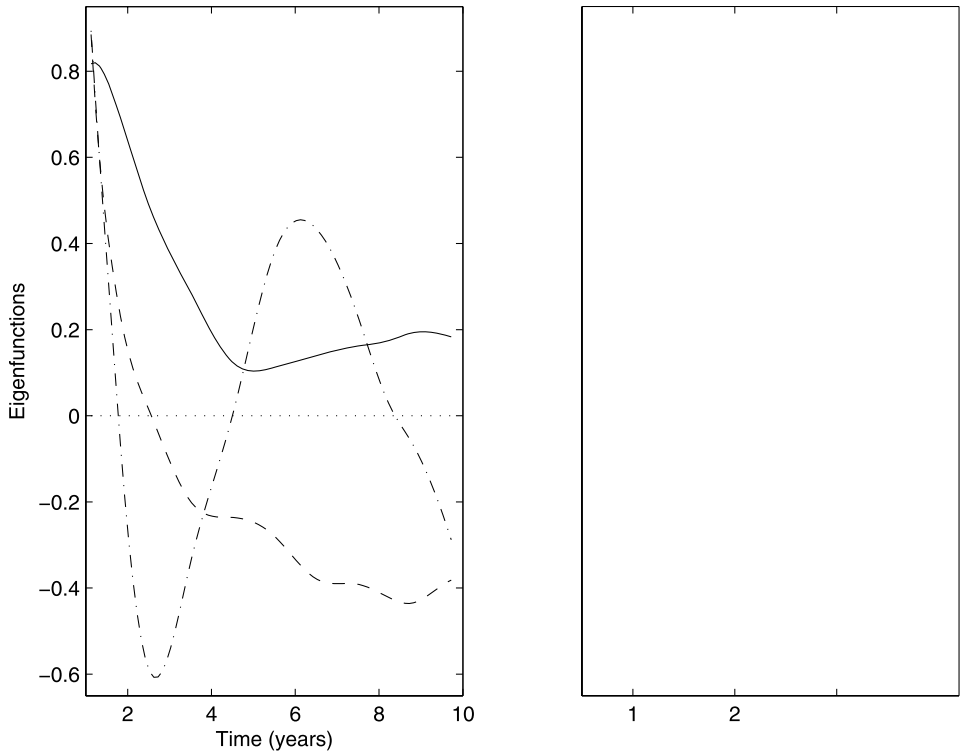
[illegible] (1958), [illegible] (1984), [illegible] (1992), [illegible] (1998) [illegible] (2005) [illegible].

[illegible] 18 ([illegible]), [illegible] 10 ([illegible]), 39 [illegible]. [illegible]: [illegible] ? [illegible] ? [illegible].

[illegible] 39 [illegible] (1954), [illegible] 10 [illegible] 15 [illegible] 10 [illegible] {1, 1.25, 1.5, 1.75, 2, 3, 4, 5, 6, 7, 8, 8.5, 9, 9.5, 10}, [illegible] $\{s_j\}_{j=1,\ldots,15}$. [illegible] $X_{ij} = (h_{i,j+1} - h_{ij})/(t_{j+1} - t_j)$, [illegible] h_{ij} [illegible] s_j [illegible] i [illegible], $t_j = (s_j + s_{j+1})/2$, $i = 1, \ldots, 39$, $j = 1, \ldots, 14$. [illegible] (2003) [illegible] (2005).

A [illegible] B [illegible].


8
10

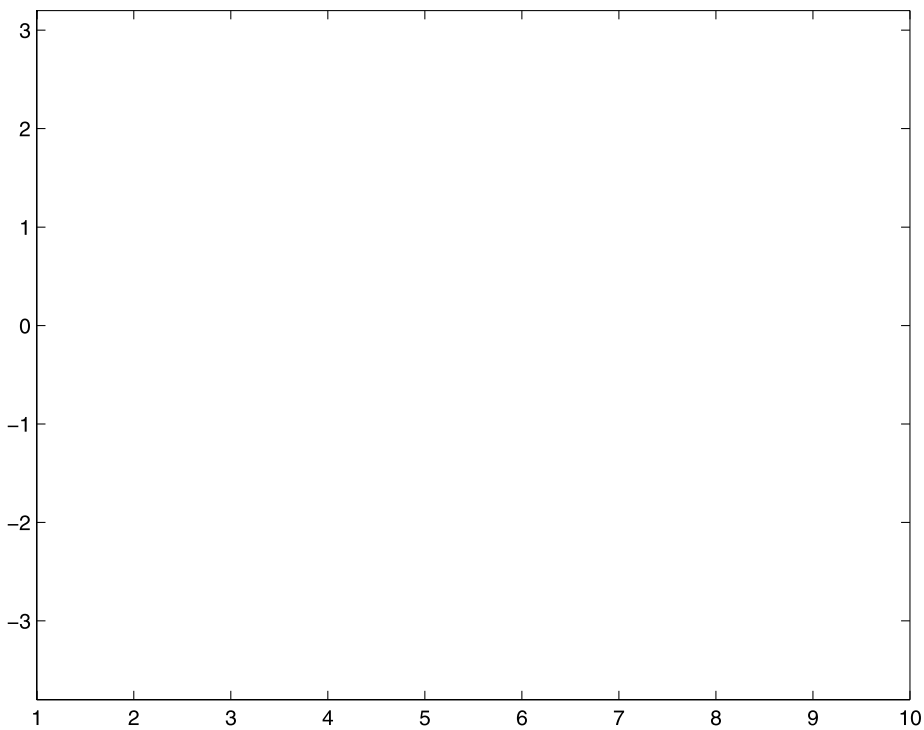

0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
Eigenfunctions
2
4
6
8
10
Time (years)
1
2

$g_i^*(t)z(t)\,dt = \sum_{j=1}^{K} \gamma_{X_i,j} z_j$, X_i z z g_i^*. g_i^* *derivative generating function* X_i. $K = 3$, z X_i z.

$$\hat{g}_i^*(t) = \sum_{j=1}^{K} \hat{\gamma}_{X_i,j} \hat{\psi}_j(t) \tag{15}$$

$\gamma_{X_i,j}$ $\psi_j(t)$ (3) (7).

$\hat{g}_i^*$ $K = 3$ 3 39 X_i .



5. Add t a e t a d r f .

5.1. *Bounds on* $P(\|X-x\| \le u)$. $P(\|X-x\| \le u)$ $u \to 0$ u. (19) $\hat{g}(x)$ $g(x)$ X (14) n. X X $X = \sum_j \xi_j \psi_j$, $(\xi_j) = \theta_j$ θ_j, $j \ge 1$, j ξ_j .

A 4. $B, \beta > 0$,

$$\theta_j = -Bj^{\beta} + o(j^{\beta}) \quad j \to \infty, \tag{16}$$

$\eta_j = \xi_j/\theta_j^{1/2}$ η,

$$\begin{aligned} &B_1 u^b \le P(|\eta| \le u) \le B_2 u^b \quad u > 0, \\ &P(|\eta| > u) \le B_3(1+u)^{-B_4} \quad u > 0, \quad B_1, \dots, B_4, b > 0. \end{aligned} \tag{17}$$

$x = 0$, b, B β (16) (17),

$$\pi(u) = \left(-\frac{b\beta}{\beta+1}\left(\frac{2}{B}\right)^{1/\beta}\right)|u|^{(\beta+1)/\beta}. \tag{18}$$

4. *If* (16) *and* (17) *hold, then, with* $\pi(u)$ *given by* (18),

$$P(\|X\| \le u) = \pi(u)^{1+o(1)} \quad \text{as } u \downarrow 0. \tag{19}$$

1 3, θ_j (16), ξ_j (17), h (12)

$\hat{g}(x)$ $g(x)$

$$h^{2\alpha} = (-2\alpha|h|)$$
$$= -\{1+o(1)\}2\alpha \left(\frac{\beta+1}{b\beta}\right)^{\beta/(\beta+1)} \left(\frac{B}{2}\right)^{1/(\beta+1)} (n)^{\beta/(\beta+1)}.$$

β, n^{-1}, β. A (16) (17) $\theta_j = (-Bj^{\beta})$, η A 4 $\{\phi_j\}$ $\beta = b = 1$ $\pi(u) = \{-c(u)^{(\beta+1)/\beta}\} = u^{-c(u)^{1/\beta}}$, $c > 0$, 0. η X.

4 $x = 0$, $P(\|X - x\| \le u)$ u, (19) $x = 0$ X $X_1 - X_2$, (19) (19), $P(\|X_1 - x\| \le u)$ x X_2. (19) $x = 0$. x, $x = \sum_j \theta_j^{1/2} x_j$ x_j x_j θ_j (16), $P(\|X - x\| \le u)$ $(-C_1 u^{-C_2})$ u 0, $(-C_1|u|^{C_2})$ 3 $C_1, C_2 > 0$. $x = 0$, (2003), A (1952). u^{C_3} $(-C_1 u^{-C_2})$ $C_3 > 0$ (2007),

5.2. *Proof of Theorem 1.* σ^2 ε (1). $N_j = \sum_i K_i(x)^j$ $j = 1, 2$, $N_2 \le K(0) N_1$, $K(\cdot)$

[illegible] $[0, c]$. [illegible],

$$
\begin{aligned}
&E[\{\hat{g}(x) - g(x)\}^2 | \mathcal{X}] \\
&\quad = [E\{\hat{g}(x)|\mathcal{X}\} - g(x)]^2 + \text{[illegible]}(\hat{g}(x)|\mathcal{X}) \\
&\quad \le \text{[illegible]}_{i=1,\dots,n} |g(X_i) - g(x)| I(\|X_i - x\| \le ch) + \frac{\sigma^2 \text{[illegible]}_i K_i^2(x)}{\{\text{[illegible]}_i K_i(x)\}^2} \\
&\quad \le \text{[illegible]}_{y:\|y\| \le ch} |g(x) - g(x+y)|^2 + \frac{\sigma^2 K(0)}{N_1}.
\end{aligned} \tag{20}
$$

[illegible] g [illegible] x [illegible], (9) [illegible] (20) [illegible] $K_i(x) \ge K_i(x) I(\|X_i - x\| \le c_1 h) \ge K(c_1) I(\|X_i - x\| \le c_1 h)$, [illegible] c_1 [illegible] (A2). [illegible], (10) [illegible] $N_1^{-1} \to 0$ [illegible] 1, [illegible] $E(N_i^{-1}) \to 0$. [illegible] (20), [illegible] (13) [illegible] (14), [illegible] (20) [illegible]

$$
\begin{aligned}
\text{[illegible]}_{g \in \mathcal{G}(C,x,\alpha)} E[\{\hat{g}(x) - g(x)\}^2 | \mathcal{X}] &\le C^2 (ch)^{2\alpha} + \frac{\sigma^2 K(0)}{N_1} \\
&\le C^2 (ch)^{2\alpha} + \frac{\sigma^2 K(0)\{1 + o_p(1)\}}{K(c_1) n P(\|X - x\| \le c_1 h)}
\end{aligned}
$$

[illegible] $E(N_1^{-1}) \le E[\{\text{[illegible]}_i I(\|X_i - x\| \le c_1 h)\}^{-1}] \asymp \{n P(\|X - x\| \le c_1 h)\}^{-1}$.

5.3. *Proof of Theorem 2.* [illegible], $x = 0$. [illegible] f [illegible] B_1, [illegible] $[-B_2, B_2]$, [illegible]. [illegible], f [illegible], B_3 [illegible] $g_1 \equiv 0$ [illegible] $g_2(y) = h^\alpha f(\|y\|/h)$. $\|y\| \le h$ [illegible], [illegible] $0 < \alpha \le 1$,

$$
\begin{aligned}
|g_2(y) - g_2(0)| &= h^\alpha |f(\|y\|/h) - f(0)| \le h^\alpha B_1 \|y\|/h \le h^\alpha B_1 (\|y\|/h)^\alpha \\
&= B_1 \|y\|^\alpha,
\end{aligned}
$$

[illegible], $\|y\| > h$,

$$
|g_2(y) - g_2(0)| \le 2h^\alpha B_3 \le 2B_3 \|y\|^\alpha.
$$

[illegible], $g_2 \in \mathcal{G}(C, 0, \alpha)$ [illegible] $(B_1, 2B_3) \le C$.
[illegible]

(21) $$P(\rho > 1) \quad 1 \quad n \to \infty.$$

,

$$2 \quad \rho = \sum_{i=1}^{n} \{g_2(X_i)^2 - 2\varepsilon_i g_2(X_i)\},$$

, $\mathcal{X}$, $s_n^2 = \sum_i g_2(X_i)^2$ $4s_n^2$. , (21)

(22) $$\lim_{B\to\infty} \lim_{n\to\infty} P(s_n^2 > B) = 0$$

2 (22).
B_2 f $0 < B \le c_1$, $|g_2(x)| \le B_3 h^\alpha I(\|x\| \le c_1 h)$,

(23) $$s_n^2 \le B_3^2 h^{2\alpha} \sum_{i=1}^{n} I(\|X_i\| \le c_1 h).$$

, $nP(\|X\| \le c_1 h) \to \infty$,

$$\frac{\sum_i I(\|X_i\| \le c_1 h)}{nP(\|X\| \le c_1 h)} \to 1$$

, (12) (23) (22).

5.4. *Proof of Theorem* 3. , $K_{i_1 i_2 j}$ $K(i_1, i_2, j|x)$. A - 3( )

(24) $$K_{i_1 i_2 j} = 0, \quad : \|X_{i_1} - x\| \le h_1, \quad \|X_{i_2} - x\| \le h_1 \quad Q_{i_1 i_2} \le h_2.$$

$\delta > 0$, $s(\delta)$ $|g(x+y) - g(x) - g_x y|$ - y $\|y\| \le \delta$. , A 3( ),

(25) $$\delta^{-1} s(\delta) \to 0 \quad \delta \downarrow 0.$$

$\mathcal{E}_{i_1 i_2}$ $\|X_{i_k} - x\| \le h_1$ $k = 1, 2$. $\mathcal{E}_{i_1 i_2}$,

$$|g(X_{i_1}) - g(X_{i_2}) - g_x(X_{i_1} - X_{i_2})| \le 2s(h_1).$$

, $\varepsilon_{i_1 i_2} = \varepsilon_{i_1} - \varepsilon_{i_2}$ $\mathcal{E}_{i_1 i_2}$,

$$|Y_{i_1} - Y_{i_2} - \{g_x(X_{i_1} - X_{i_2}) + \varepsilon_{i_1 i_2}\}| \le 2s(h_1).$$

$\xi_{i_1 i_2 j} = \xi_{i_1 j} - \xi_{i_2 j}$, $g_x(X_{i_1} - X_{i_2}) = \sum_k \xi_{i_1 i_2 k}\gamma_{xk}$, (24),

$$\begin{aligned}
&\left\| \sum_{i_1,i_2}^{(j)} (Y_{i_1} - Y_{i_2}) K_{i_1 i_2 j} \right. \\
&\quad \left. - \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j} \sum_{k=1}^{\infty} \xi_{i_1 i_2 k}\gamma_{xk} + \sum_{i_1,i_2}^{(j)} \varepsilon_{i_1 i_2} K_{i_1 i_2 j} \right\| \\
&\quad \le 2s(h_1) \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j}.
\end{aligned} \tag{26}$$

$$\begin{aligned}
|\hat{\xi}_{i_1 i_2 j} - \xi_{i_1 i_2 j}| &= \left| (X_{i_1} - X_{i_2})(\hat{\psi}_j - \psi_j) \right| \\
&\le \|X_{i_1} - X_{i_2}\| \|\hat{\psi}_j - \psi_j\| \le 2h_1 \|\hat{\psi}_j - \psi_j\|,
\end{aligned} \tag{27}$$

$\varepsilon_{i_1 i_2}$ (24), (26) (27),

$$\begin{aligned}
&\left\| \sum_{i_1,i_2}^{(j)} (Y_{i_1} - Y_{i_2}) K_{i_1 i_2 j} \right. \\
&\quad - \gamma_{xj} \sum_{i_1,i_2}^{(j)} \hat{\xi}_{i_1 i_2 j} K_{i_1 i_2 j} \\
&\quad \left. + \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j} \sum_{k:k\neq j} \xi_{i_1 i_2 k}\gamma_{xk} + \sum_{i_1,i_2}^{(j)} \varepsilon_{i_1 i_2} K_{i_1 i_2 j} \right\| \\
&\quad \le 2\{s(h_1) + |\gamma_{xj}| h_1 \|\hat{\psi}_j - \psi_j\|\} \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j}.
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\left\| \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j} \sum_{k:k\neq j} \xi_{i_1 i_2 k}\gamma_{xk} \right\| \\
&\quad = \left\| \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j} \sum_{k:k\neq j} \gamma_{xk} (X_{i_1} - X_{i_2})\psi_k \right\| \\
&\quad \le \sum_{i_1,i_2}^{(j)} K_{i_1 i}
\end{aligned}$$

$$\leq \|g_x\| \square^{(j)} K_{i_1,i_2}$$

[illegible] (32) [illegible] $\hat{\gamma}_{xj}$ [illegible] (7). [illegible] (27) [illegible]

$$\sum_{i_1,i_2}^{(j)} \hat{\xi}_{i_1 i_2 j} K_{i_1 i_2 j} \geq \sum_{i_1,i_2}^{(j)} (0, \xi_{i_1 j} - \xi_{i_2 j} - 2h_1 \|\hat{\psi}_j - \psi_j\|) K_{i_1 i_2 j}$$

$$\geq \sum_{i_1,i_2}^{(j)} (0, \xi_{i_1 j} - \xi_{i_2 j}) K_{i_1 i_2 j} - 2h_1 \|\hat{\psi}_j - \psi_j\| \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j}. \tag{33}$$

[illegible] $\sum_{i_1,i_2}^{(j)}$ [illegible] (i_1, i_2) [illegible] $\hat{\xi}_{i_1 i_2 j} > 0$. [illegible] A [illegible] 4( ), ( ) [illegible] ( ), [illegible] $B > 0$,

$$\sum_{i_1,i_2}^{(j)} (0, \xi_{i_1 j} - \xi_{i_2 j}) K_{i_1 i_2 j} \geq \{1 + o_p(1)\} B h_1 \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j}. \tag{34}$$

[illegible], A [illegible] 3( ), $n^{-1/2}/$ [illegible] $(h_1, h_2) \to 0$. A [illegible] 3( ) [illegible] ( ), [illegible]

$$\|\hat{\psi}_j - \psi_j\| = O_p(n^{-1/2}). \tag{35}$$

[illegible], (33) (35) [illegible]

$$\sum_{i_1,i_2}^{(j)} \hat{\xi}_{i_1 i_2 j} K_{i_1 i_2 j} \geq \{1 + o_p(1)\} B h_1 \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j} \tag{36}$$

[illegible] B [illegible] (34). [illegible] (7). (7), (25), (32) [illegible] (36), [illegible]

$$\hat{\gamma}_{xj} = \gamma_{xj} + O_p \left(\frac{\sum_{i_1,i_2}^{(j)} \varepsilon_{i_1 i_2} K_{i_1 i_2 j}}{h_1 \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j}} \right) + o_p(1). \tag{37}$$

[illegible] (37), [illegible] X_i, [illegible]

$$O_p \left(h_1^2 \sum_{i_1,i_2}^{(j)} K_{i_1 i_2 j} \right)^{-1} = O_p[\{(nh_1)^2 E(k_{i_1 i_2 j})\}^{-1}] = o_p(1),$$

[illegible], [illegible] A [illegible] 3( ). [illegible], (37) [illegible] $\hat{\gamma}_{xj} = \gamma_{xj} + o_p(1)$, [illegible] 3.

5.5. *Proof of Theorem 4.* [illegible] $t \in (0,1)$ [illegible] $D_t = (\sum_j \theta_j^{1-t})^{-1}$,

$$P(\|X\| \le u) = P\left(\sum_{j=1}^{\infty} \theta_j \eta_j^2 \le u^2\right) \begin{cases} \le \prod_{j=1}^{\infty} P(\theta_j \eta_j^2 \le u^2), \\ \ge \prod_{j=1}^{\infty} P(\theta_j^t \eta_j^2 \le D_t u^2), \end{cases} \tag{38}$$

[illegible]

$$\begin{aligned} P\left(\sum_{j=1}^{\infty} \theta_j \eta_j^2 \le u^2\right) &= P\left(\sum_{j=1}^{\infty} \theta_j^{1-t}(\theta_j^t \eta_j^2 - D_t u^2) \le 0\right) \\ &\ge P(\theta_j^t \eta_j^2 \le D_t u^2 \text{ [illegible] } j). \end{aligned}$$

[illegible] $J = J(u)$ [illegible] $u/\theta_j^{1/2} \le \zeta$, [illegible] ζ [illegible] $B_1 u^b \le P(|\eta| \le u) \le B_2 u^b$ [illegible] $0 \le u \le \zeta$. [illegible],

$$\begin{aligned} \prod_{j=1}^{\infty} P(\theta_j \eta_j^2 \le u^2) &\le \prod_{j=1}^{J} P(|\eta| \le u\theta_j^{-1/2}) \\ &= u^{bJ} \exp\left(\frac{1}{2} bB \sum_{j=1}^{J} j^{\beta} + o(J^{\beta+1})\right) \\ &= \exp\left(-\frac{bB\beta}{2(\beta+1)} J^{\beta+1} + o(J^{\beta+1})\right) \\ &= \pi(u)^{1+o(1)} \end{aligned} \tag{39}$$

[illegible] $u \downarrow 0$, [illegible] π [illegible] (18).

[illegible] J [illegible] $D_t^{1/2} u/\theta_j^{t/2} \le \zeta$. [illegible] (39), [illegible]

$$\begin{aligned} \prod_{j=1}^{J} P(\theta_j^t \eta_j^2 \le D_t u^2) &= \exp\left(-\frac{b\beta}{\beta+1}\left(\frac{2}{Bt}\right)^{1/\beta} |\text{[illegible]}| u|^{(\beta+1)/\beta} + o\left(|\text{[illegible]}| u|^{(\beta+1)/\beta}\right)\right) \\ &= \pi(u)^{t^{-1/\beta}+o(1)}. \end{aligned} \tag{40}$$

A [illegible], [illegible] $j \ge J+1$,

π

$B_5 = B_5(t) \in (0,1)$, $\pi_j \in (0, B_5)$ $j \geq J+1$,

$$1 - \pi_j = \exp\left(-\sum_{k=1}^{\infty} \frac{\pi_j^k}{k}\right) \geq \exp(-B_6 \pi_j)$$

$$\sum_{j=J+1}^{\infty} (1-\pi_j) \geq \exp\left(-B_6 \sum_{j=J+1}^{\infty} \pi_j\right) \geq \exp\left(-B_7 \sum_{j=J+1}^{\infty} (\theta_j^{t/2}/u)^{B_4}\right),$$

(40). (40), $t \in (0,1)$ (40), 1, , $u \downarrow 0$,

$$\sum_{j=1}^{\infty} P(\theta_j^t \eta_j^2 \leq D_t u^2) = \pi(u)^{1+o(1)}. \tag{42}$$

, (38), (39) (42) (19).

Acknowledgment. A

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