Functional Additive Models

1.

functional additive model ( )

Drosophila

Y ζ_m β K M

3.

ξ_{ik} ( ) ( ) ( ) ( )

$$E(Y-\mu_Y|\xi_k)=b_k\xi_k \qquad E(\zeta_m|\xi_k)=b_{km}\xi_k \qquad ( )$$

( ) ( ) ( ) $b_k\xi_k$ $b_{km}\xi_k$ ( ) ( ) ( ) ξ_k $f_k(\xi_k)$ $f_k(\cdot)$, $k=\ ,\ ,\ldots$ $f_{km}(\cdot)$, $k,m=\ ,\ ,\ldots$, ( ) ( ) ξ_k

$$E(Y|X)=\mu_Y+\sum_{k=}^{\infty} f_k(\xi_k) \qquad ( )$$

$$E(Y(t)|X)=\mu_Y(t)+\sum_{k=}^{\infty}\sum_{m=}^{\infty} f_{km}(\xi_k)\psi_m(t), \qquad ( )$$

$$\begin{aligned} &Ef_k(\xi_k)=\ ,\quad k=\ ,\ ,\ldots,\\ &Ef_{km}(\xi_k)=\ ,\quad k=\ ,\ ,\ldots,m=\ ,\ ,\ldots. \end{aligned} \qquad ( )$$

Y ξ_k ξ_k ( ) ( )

$$\begin{aligned} E(Y-\mu_Y|\xi_k) &= E\{E(Y-\mu_Y|X)|\xi_k\}\\ &= E\left\{\sum_{j=}^{\infty} f_j(\xi_j)\Big|\xi_k\right\} = f_k(\xi_k), \end{aligned} \qquad ( )$$

$$\begin{aligned} E(\zeta_m|\xi_k) &= E\{E(\zeta_m|X)|\xi_k\}\\ &= E\left\{\sum_{j=}^{\infty} f_{jm}(\xi_j)\Big|\xi_k\right\} = f_{km}(\xi_k). \end{aligned} \qquad ( )$$

$f_k(\xi_k)=E(Y-\mu_Y|\xi_k)$ $f_{km}(\xi_k)=E(\zeta_m|\xi_k)$, $k,m=\ ,\ ,\ldots,$ ( ) (,) ( ) ( ) f_k f_{km} ( ) ( )

4.

$\hat{\varphi}_k$ $\hat{\lambda}_k$ $\hat{\xi}_{ik}$, *principal analysis by conditional expectation* (E) K ( ) ( ). E

$\hat{f}_k$ $\hat{f}_{km}$ f_k f_{km} , f_k $\{\hat{\xi}_{ik}, Y_i\}_{i=\ ,\ldots,n}$, $\hat{\xi}_{ik}$ ( ) ,

$$\sum_{i=}^{n} K\left(\frac{\hat{\xi}_{ik}-x}{h_k}\right)\{Y_i-\beta\ -\beta\ (x-\hat{\xi}_{ik})\} \qquad ( )$$

β β $\hat{f}_k(x)=\hat{\beta}\ (x)-\bar{Y}$, h_k K f_{mk}

$\{\hat{\xi}_{ik}, \hat{\zeta}_{im}\}_{i= ,\dots,n}$ (,).

$$\sum_{i=}^{n} K\left(\frac{\hat{\xi}_{ik}-x}{h_{mk}}\right)\{\hat{\zeta}_{im}-\beta - \beta (x-\hat{\xi}_{ik})\} \qquad ( )$$

β β $\hat{f}_{mk}(x)=\hat{\beta} (x)$. h_{mk} ( )

$$\widehat{E}(Y|X)=\bar{Y}+\sum_{k=}^{K}\hat{f}_k(\xi_k). \qquad ( )$$

R ( ).

$$R = -\frac{\sum_{i=}^{n}\{Y_i-E(Y_i|X_i)\}}{\sum_{i=}(Y_i-\mu_Y)}$$
$$\widehat{R} = -\frac{\sum_{i=}^{n}\{Y_i-\widehat{E}(Y_i|X_i)\}}{\sum_{i=}(Y_i-\bar{Y})}, \qquad ( )$$

$E(Y_i|X_i)$ $\widehat{E}(Y_i|X_i)$ i ( ) ( ),

( ).

$$\widehat{E}\{Y(t)|X\}=\hat{\mu}_Y(t)+\sum_{m=}^{M}\sum_{k=}^{K}\hat{f}_{mk}(\xi_k)\hat{\psi}_m(t), \qquad t\in\mathcal{T}, \qquad ( )$$

$$R = -\frac{\sum_{i=}^{n}\int [Y_i(t)-E\{Y_i(t)|X_i\}] \, dt}{\sum_{i=}^{n}\int \{Y_i(t)-\mu_Y(t)\} \, dt},$$
$$\widehat{R} = -\frac{\sum_{i=}^{n}\sum_{l=}^{m_i}[V_{il}-\widehat{E}\{Y_i(t_{il})|X_i\}] (t_{il}-t_{i,l-})}{\sum_{i=}^{n}\sum_{l=}^{m_i}\{V_{il}-\mu_Y(t_{il})\} (t_{il}-t_{i,l-})} \qquad ( )$$

$E\{Y_i(t)|X_i\}$ $\widehat{E}(Y_i(t)|X_i)$ i ( ) ( ). t_{il}

5. A

E ξ_{ik} $\hat{\xi}_{ik}$, ζ_{im} $\hat{\zeta}_{im}$, $k=\ ,\dots,K$, $m=\ ,\dots,M$. f_k f_{km}

(,) (,) (,).

$|\hat{\xi}_{ik}-\xi_{ik}|$ $|\hat{\zeta}_{im}-\zeta_{im}|$. i i, $i=\ ,\dots,n$ ( ), f_k ( ) f_{mk} ( )

$\{\hat{\xi}_{ik}, Y_i\}$ $\{\hat{\xi}_{ik}, \hat{\zeta}_{im}\}$ $i=\ ,\dots,n$, $n\to\infty$. K M $K=K(n)\to\infty$ $M=M(n)\to\infty$ ( ) ( ) ( ) ( ) ( )

Theorem 1. (,) ( ), (,), (,), (,), (,) ( ). $k\geq$ $\lambda_j, j\leq k$

$$\hat{f}_k(x)-f_k(x)\xrightarrow{p} \ , \qquad ( )$$

( ), (,) ( ) (,). k m $\lambda_j, j\leq k$ $\rho_l, l\leq m$

$$\hat{f}_{km}(x)-f_{km}(x)\xrightarrow{p} \ , \qquad ( )$$

( ),

$\tilde{\theta}_k(x)=|\hat{f}_k(x)-f_k(x)|$ $\tilde{\vartheta}_{mk}(x)=|\hat{f}_{mk}(x)-f_{mk}(x)|$ ( ),

Theorem 2. (,) ( ), ( ), ( ), (,), (,), (,) (,).

$$\widehat{E}(Y|X)-E(Y|X)\xrightarrow{p} \ , \qquad ( )$$

$\widehat{E}(Y|X)=\bar{Y}+\sum_{k=}^{K}\hat{f}_k(\xi_k)$ ( ), (,) ( ), ( ), ( ), (,). $t\in\mathcal{T}$,

$$\widehat{E}\{Y(t)|X\}-E\{Y(t)|X\}\xrightarrow{p} \ , \qquad ( )$$

$\widehat{E}\{Y(t)|X\}=\hat{\mu}_Y(t)+\sum_{k=}^{K}\sum_{m=}^{M}\hat{f}_{mk}(\xi_k)\hat{\psi}_m(t)$. $\hat{f}_{mk}(\xi_k)$ ( ),

$\theta_n^*=|\widehat{E}(Y|X)-E(Y|X)|$ $\vartheta_n^*=|\widehat{E}(Y(t)|X)-E(Y(t)|X)|$ ( )

6. A

( ) ( ) $n=$ X_i, $\mu_X(s)=s+\ (s)$, $\leq s\leq$ $\varphi (s)=-\ (\pi s/\ )/\sqrt{\ }$ $\varphi (s)=\ (\pi s/\ )/\sqrt{\ }$, $\leq s\leq$ $\lambda =$ $\lambda =$ $\lambda_k=\ $, $k\geq$ (,) $_{ij}\sim$ (, .), ( ) $\xi_{ik}\sim\mathcal{N}(\ ,\lambda_k)$ ( ) $\xi_{ik}=\sqrt{\lambda_k}(Z_{ik}-\ )/$ $Z_{ik}\sim$ (,). $k=\ ,$

$[\,,\,]$

$\{\,,\dots,\,\}$

$\{\,,\dots,\,\}$

X_i^* U_{ij}^* U_{ij}

Y_i^*

$Y_i = \sum_{k=} f_k(\xi_{ik}) + \varepsilon_i$ f_k ξ_{ik} X_i Y_i $\varepsilon_i \sim (\,,.\,)$ $\mu_Y =$ $\sigma_Y = .$

$f_k(x) = x - \lambda_k$ $k = \,,$ $f\,(x) = x$ $f\,(x) = x/$ f_k

$\hat{f}_k(x) = \sum_{i=}^{n} (Y_i - \bar{Y})(\hat{\xi}_{ik} - \bar{\hat{\xi}}_{\cdot k})\hat{\lambda}_k^{-} x$ $\bar{Y}_i = \sum_{i=}^{n} Y_i/n$ $\bar{\hat{\xi}}_{\cdot k} = \sum_{i=}^{n} \hat{\xi}_{ik}/n$ $\hat{\lambda}_k$ k X $\hat{\xi}_{ik}$

f_k

f_{km} $\hat{f}_{km}(x) = \sum_{i=}^{n} (\hat{\zeta}_{im} - \bar{\zeta}_{\cdot m})(\hat{\xi}_{ik} - \bar{\hat{\xi}}_{\cdot k})\hat{\lambda}_k^{-} x$

$Y_i(t) = \mu_Y(t) + \zeta_i\, \psi\,(t)$ $\mu_Y(t) = t + \,(t)$ $\psi\,(t) = -\,(\pi t/\,)/\sqrt{\,}$ ζ_i $\le t \le$ ζ_i

$f_{km}, k =\,, \,, m =$ $f_k\,(x) = x - \lambda_k$ $k = \,,$ $\zeta_i = \sum_{k=} (\xi_{ik} - \lambda_k)$ $f\,(x) = x$ $f\,(x) = x/$ $\zeta_i = \xi_i + \xi_i\,/$ ε_{il} $\mathcal{N}(\,,.\,)$

X_i^* Y_i^* U_{ij}^* V_{il}^* U_{ij} V_{il}

$$\mathrm{E}_i = \frac{(Y_i^* - \hat{Y}_i^*)}{Y_i^*}$$

$$\mathrm{E}_{i,f} = \frac{\int (Y_i^*(t) - \hat{Y}_i^*(t))\, dt}{\int Y_i^*\,(t)\,dt},$$

f_k f_{km}

$n =$

7. A A X A A

Drosophila

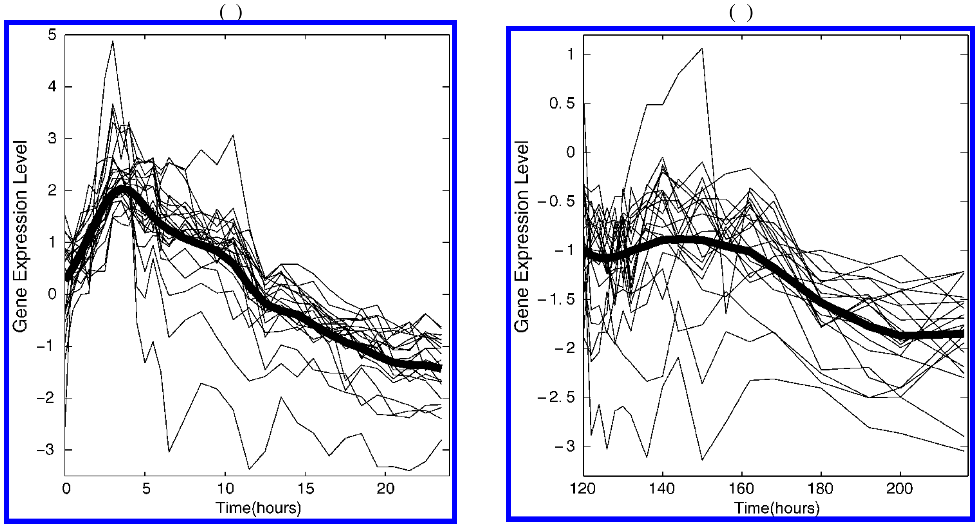
$n =$

$\hat{\zeta}_i$, $\hat{\zeta}_i$, $\hat{\zeta}_i$ $\hat{\zeta}_i$

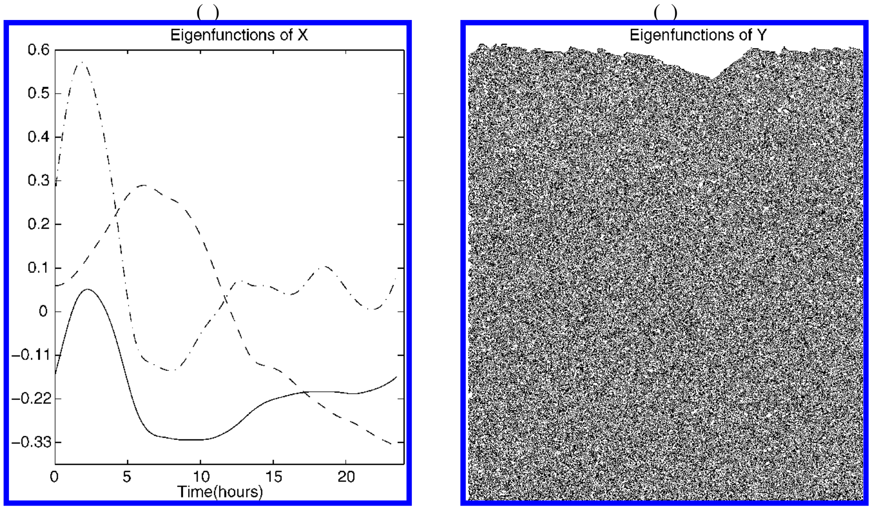
$\hat{\xi}_i$, $\hat{\xi}_i$ $\hat{\xi}_i$

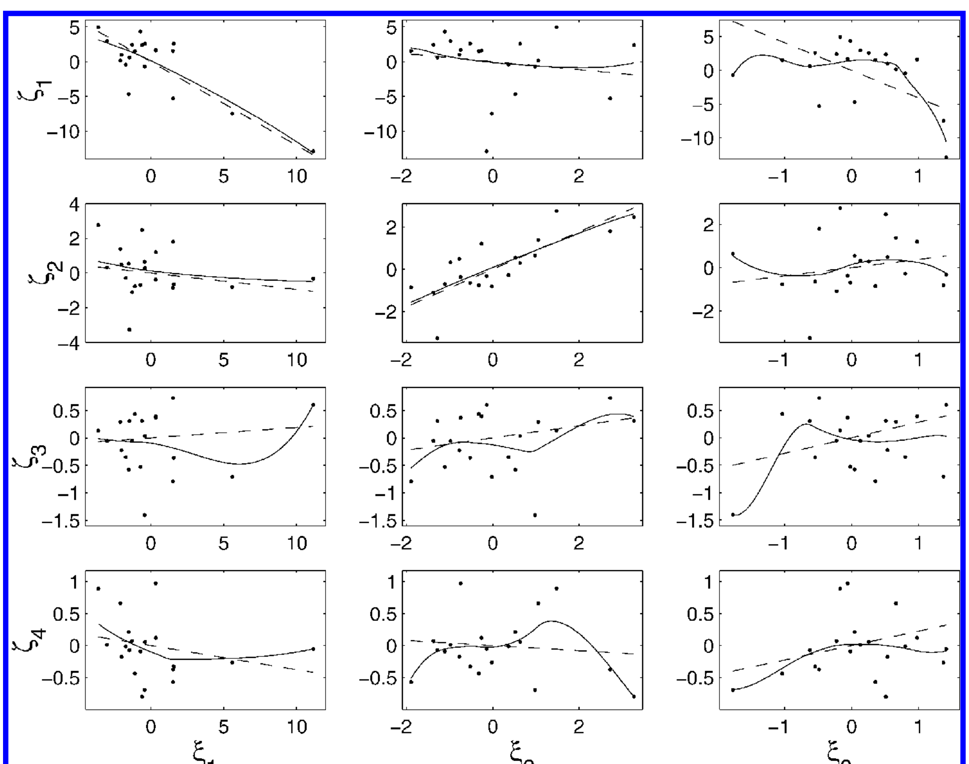
$\hat{\zeta}_i$ $\hat{\xi}_i$

( )



( ) ( )


Eigenfunctions of X
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.11
−0.22
−0.33
0
5
10
15
20
Time(hours)
Eigenfunctions of Y


ζ1
ζ2
ζ3
ζ4
ξ1
ξ2
ξ3

R ( ).

$\mathrm{E}_{(-i),f}$ ( ).

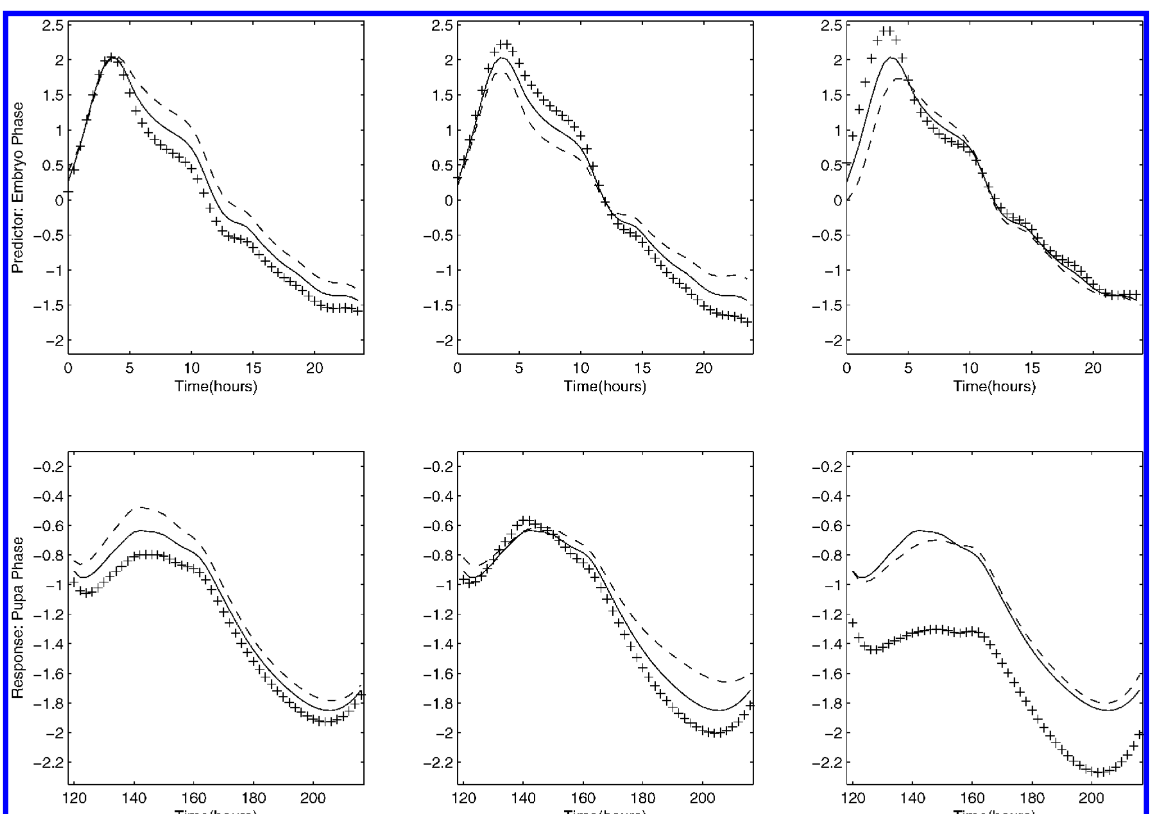
				R

E ( ).

$\mathrm{E}_{(-i),f}$,

R ( )

8. A



$\hat{\mu}_X(s)+\alpha\hat{\varphi}_k(s)$. $\alpha =$ ( ). $\alpha =$ (−−) $\alpha = -$ (++) $k =$ ( ). $k =$ ( ). $k =$ ( ). $\hat{\mu}_Y(t)+\sum_{m=}^{M}\{\hat{f}_{km}(\alpha)+\sum_{\neq k}\hat{f}_{m}(\;)\}\hat{\psi}_m(t)$.

A X A

A A

A.1

$\varphi_k, k = \ , \ ,\ldots,$ X $(A_{G_X} f)(s) = \lambda f(s)$ $f \in L\ (\mathcal{S})$ $(A_{G_X} f)(t) = \int f(s) \times G_X(s,t)\,ds$ $\int \varphi_j(s)\varphi_k(s)\,ds = \delta_{jk}$ $\delta_{jk} =$ $j = k$ $=$ $j \neq k$ φ_j $\lambda \ \geq \lambda \ \geq \cdots.$ Y ψ_m ρ_m $\{\mu_X, \varphi_j\}$ $\{\mu_Y, \psi_k\}$ $G_X(s\ ,s\ ) = \sum_k \lambda_k \times \varphi_k(s\ )\varphi_k(s\ )$ $G_Y(t\ ,t\ ) = \sum_m \rho_m \psi_m(t\ )\psi_m(t\ )$ X Y K X Y

$$X(s) = \mu_X(s) + \sum_{j=}^{\infty} \xi_j \varphi_j(s)$$

$$Y(t) = \mu_Y(t) + \sum_{k=}^{\infty} \zeta_k \psi_k(t).$$

$\{\varphi_k, k = \ , \ ,\ldots\}$ $\{\psi_m, m = \ , \ ,\ldots\}$ β_k β_{km} $\beta(s) = \sum_{k=}^{\infty} \beta_k \varphi_k(t)$ $\beta(s,t) = \sum_{k=}^{\infty} \sum_{m=}^{\infty} \beta_{km}\varphi_k(s)\psi_m(t)$

$$U_{ij} = X_i(s_{ij}) + \ _{ij}$$
$$= \mu_X(s_{ij}) + \sum_{k=}^{\infty} \xi_{ik}\varphi_k(s_{ij}) + \ _{ij},$$
$$s_{ij} \in \mathcal{S}, \quad \leq i \leq n, \quad \leq j \leq n_i,$$

$$V_{il} = Y_i(t_{il}) + \varepsilon_{il}$$
$$= \mu_Y(t_{il}) + \sum_{m=}^{\infty} \zeta_{im}\psi_k(t_{il}) + \varepsilon_{il},$$
$$s_{il} \in \mathcal{T}, \quad \leq i \leq n, \quad \leq l \leq m_i.$$

$\mathbf{U}_i = (U_i\ ,\ldots,U_{in_i})^T$ $\boldsymbol{\mu}_{X_i} = (\mu_X(s_i\ ),\ldots,\mu_X(s_{in_i}))^T$ $\boldsymbol{\varphi}_{ik} = (\varphi_k(s_i\ ),\ldots,\varphi_k(s_{in_i}))^T$ ξ_{ik} $\lambda_k \boldsymbol{\varphi}_{ik}^T \ _{U_i}^{-} (\mathbf{U}_i - \boldsymbol{\mu}_{X_i})$ (j,l) $n_i \times n_i$ $_{U_i}$ $(\ _{U_i})_{j,l} = G_X(s_{ij}, s_{il}) + \sigma_X \delta_{jl}$ $\delta_{jl} =$ $j = l$ $\delta_{jl} =$ $j \neq l$ ξ_{ik} $\boldsymbol{\mu}_{X_i}$ λ_k $\boldsymbol{\varphi}_{ik}$ X_i

$$\hat{\xi}_{ik} = \hat{\lambda}_k \hat{\boldsymbol{\varphi}}_{ik}^T \widehat{\ }_{U_i}^{-} (\mathbf{U}_i - \hat{\boldsymbol{\mu}}_{X_i}),$$

(j,l) $\widehat{\ }_{U_i}$ $(\widehat{\ }_{U_i})_{j,l} = \widehat{G}_X(s_{ij}, s_{il}) + \hat{\sigma}_Y(s_{ij})\delta_{jl}$ E $\hat{\xi}_{ik}, \hat{\zeta}_{im}$

$$\hat{\xi}_{ik}^I = \sum_{j=}^{n_i} (U_{ij} - \hat{\mu}_X(t_{ij}))\hat{\varphi}_k(s_{ij})(s_{ij} - s_{i,j-}\ ),$$

$$\hat{\zeta}_{im}^I = \sum_{j=}^{n_i} (V_{ij} - \hat{\mu}_Y(t_{ij}))\hat{\psi}_k(t_{ij})(t_{ij} - t_{i,j-}\ ),$$

$\xi_{ik} = \int\{X_i(s) - \mu_X(s)\}\varphi_k(s)\,ds$ $\zeta_{im} = \int\{Y_i(t) - \mu_Y(t)\}\psi_m(t)\,dt$ E

A.2

E μ_X

$$\sum_{i=}^{n} \sum_{j=}^{n_i} K\left(\frac{s_{ij} - s}{b_X}\right)\{U_{ij} - \beta^X - \beta^X(s - s_{ij})\}$$

β^X β^X $\hat{\mu}_X(s) = \hat{\beta}^X(s)$ K b_X E $G_X(s\ ,s\ )$ X $^X_i(s_{ij}\ ,s_{ij}\ ) = (U_{ij}\ - \hat{\mu}_X(s_{ij}\ ))(U_{ij}\ - \hat{\mu}_X(s_{ij}\ ))$

$$\sum_{i=}^{n} \sum_{\leq j \neq j \leq n_i} K\left(\frac{s_{ij} - s}{h_X}, \frac{s_{ij} - s}{h_X}\right)$$
$$\times \left\{\ ^X_i(s_{ij}\ ,s_{ij}\ ) - f\left(\beta^X, (s\ ,s\ ), (s_{ij}\ ,s_{ij}\ )\right)\right\},$$

$f(\beta^X, (s\ ,s\ ), (s_{ij}\ ,s_{ij}\ )) = \beta^X + \beta^X(s\ - s_{jl}\ ) + \beta^X(s\ - s_{ij}\ )$ $\boldsymbol{\beta}^X = (\beta^X, \beta^X, \beta^X)^T$ $\widehat{G}_X(s\ ,s\ ) = \hat{\beta}^X(s\ ,s\ )$ K h_X $j\ = j$ σ_X G_X $\widetilde{G}_X(s)$ $\{\widehat{G}_X(s,s) + \sigma_X\}$ $\widehat{V}_X(s)$

h^*_X, $a = \{s \; s \in \mathcal{S}\}$, $b = \{s \; s \in \mathcal{S}\}$, $|\mathcal{S}| = b - a$, $\mathcal{S} = [a + |\mathcal{S}|/ \; , b - |\mathcal{S}|/ \;]$ σ_X

$$\hat{\sigma}_X = \int \{\widehat{V}_X(s) - \widetilde{G}_X(s)\} ds/|\mathcal{S}| \qquad (\;)$$

$\hat{\sigma}_X >$ $\hat{\sigma}_X =$ E

$\{\lambda_k, \varphi_k\}_{k\geq}$ $\{\hat{\lambda}_k,$

ξ_k $t \in \mathcal{T}$

( ) $\sum_{k=}^{K} [|f_k''(\xi_k)|h_k + n^{-\ /} \{ (Y|\xi_k)\}^{\ /} p_k^{-\ /}(\xi_k) \times h_k^{-\ /}] \to$

( ) $\sum_{k=}^{K} \sum_{m=}^{M} [|f_{mk}''(\xi_k)\psi_m(t)|h_{mk} + n^{-\ /} \{ (\zeta_m|\xi_k)\}^{\ /} \times p_k^{-\ /}(\xi_k)|\psi_m(t)|h_{mk}^{-\ /}] \to$

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Supplement for "Functional Additive Models"

1. NOTATIONS AND AUXILIARY RESULTS

Covariance operators are denoted by $\mathcal{G}_X$, $\widehat{\mathcal{G}}_X$, generated by kernels G_X, $\widehat{G}_X$, i.e., $\mathcal{G}_X(f) = \int_{\mathcal{T}} G_X(s,t) f(s)\,ds$, $\widehat{\mathcal{G}}_X(f) = \int_{\mathcal{T}} \widehat{G}_X(s,t) f(s)\,ds$ for any $f \in L^2(\mathcal{T})$. Define

$$D_X = \int_{\mathcal{T}^2} \{\widehat{G}_X(s,t) - G_X(s,t)\}^2\,ds\,dt, \qquad \delta_k^x = \min_{1\le j\le k}(\lambda_j - \lambda_{j+1}),$$

$$K_0 = \inf\{j \ge 1 : \lambda_j - \lambda_{j+1} \le 2D_X\} - 1, \qquad \pi_k^x = 1/\lambda_k + 1/\delta_k^x. \tag{1}$$

Let $K = K(n)$ denote the numbers of leading eigenfunctions included to approximate X as sample size n varies, i.e., $\widehat{X}_i(s) = \widehat{\mu}_X(s) + \sum_{k=1}^{K} \widehat{\xi}_{ik}\widehat{\phi}_k(s)$. Analogously, define the quantities $\mathcal{G}_Y$, $\widehat{\mathcal{G}}_Y$, D_Y, δ_m^y, π_m^y, M_0 and M for the process Y, for the case of functional responses. The following lemma establishes the uniform convergence rates for the estimators of the FPCs. This serves as the basis for the subsequent developments in the proofs in Section 2.

Lemma 1 *Under (A1)–(A5), (C1.1)–(C1.2) and (C2.1),*

$$\sup_{t\in\mathcal{S}} |\widehat{\mu}_X(s) - \mu_X(s)| = O_p\Big(\frac{1}{\sqrt{n}b_X}\Big), \quad \sup_{s_1,s_2\in\mathcal{S}} |\widehat{G}_X(s_1,s_2) - G_X(s_1,s_2)| = O_p\Big(\frac{1}{\sqrt{n}h_X^2}\Big), \tag{2}$$

and as a consequence, $\widehat{\sigma}_X^2 - \sigma_X^2 = O_p(n^{-1/2}h_X^{-2} + n^{-1/2}h_X^{*-1})$. Considering eigenvalues λ_k of multiplicity one, $\widehat{\phi}_k$ can be chosen such that

$$P\Big(\sup_{1\le k\le K_0} |\widehat{\lambda}_k - \lambda_k| \le D_X\Big) = 1, \quad \sup_{s\in\mathcal{S}} |\widehat{\phi}_k(s) - \phi_k(s)| = O_p\Big(\frac{\pi_k^x}{\sqrt{n}h_X^2}\Big), \quad k = 1,\ldots,K_0, \tag{3}$$

where D_X, π_k^x and K_0 are defined in (1).

Analogously, under (B1)–(B5), (C1.1)–(C1.2) and (C2.2),

$$\sup_{t\in\mathcal{T}} |\widehat{\mu}_Y(t) - \mu_Y(t)| = O_p\Big(\frac{1}{\sqrt{n}b_Y}\Big), \quad \sup_{t_1,t_2\in\mathcal{T}} |\widehat{G}_Y(t_1,t_2) - G_Y(t_1,t_2)| = O_p\Big(\frac{1}{\sqrt{n}h_Y^2}\Big), \tag{4}$$

and as a consequence, $\widehat{\sigma}_Y^2 - \sigma_Y^2 = O_p(n^{-1/2}h_Y^{-2} + n^{-1/2}h_Y^{*-1})$. Considering eigenvalues ρ_m of multiplicity one, $\widehat{\psi}_m$ can be chosen such that

$$P\Big(\sup_{1\le m\le M_0} |\widehat{\rho}_m - \rho_m| \le D_Y\Big) = 1, \quad \sup_{t\in\mathcal{T}} |\widehat{\psi}_m(t) - \psi_m(t)| = O_p\Big(\frac{\pi_k^y}{\sqrt{n}h_Y^2}\Big), \quad m = 1,\ldots,M_0, \tag{5}$$

where $D_{\mathbf{Y}}$, $\pi^{\mathbf{y}}_{\mathbf{k}}$ *and* M_0 *are defined analogously to* *for process* Y.

Denote $\|f\|_\infty = \sup_{x\in\mathcal{A}}|f(t)|$ for any function f with support $\mathcal{A}$, and $\|g\| = \sqrt{\int_{\mathcal{A}} g^2(t)dt}$ for any $g \in L^2(\mathcal{A})$, and define

$$
\begin{aligned}
\theta^{(1)}_{\mathbf{ik}} &= c_1\|X_{\mathbf{i}}\| + c_2\|X_{\mathbf{i}}X'_{\mathbf{i}}\|_\infty \; {}^*_{\mathbf{X}} + c_3, & Z^{(1)}_{\mathbf{k}} &= \sup_{s\in\mathcal{S}}|\phi_{\mathbf{k}}(s) - \phi_{\mathbf{k}}(s)|,\\
\theta^{(2)}_{\mathbf{ik}} &= 1 + \|\phi_{\mathbf{k}}\phi'_{\mathbf{k}}\|_\infty \; {}^*_{\mathbf{X}}, & Z^{(2)}_{\mathbf{k}} &= \sup_{s\in\mathcal{S}}|\mu_{\mathbf{X}}(s) - \mu_{\mathbf{X}}(s)|,\\
\theta^{(3)}_{\mathbf{ik}} &= c_4\|X_{\mathbf{i}}\|_\infty + c_5\|X'_{\mathbf{i}}\|_\infty + c_6, & Z^{(3)}_{\mathbf{k}} &= \|\phi'_{\mathbf{k}}\|_\infty \; {}^*_{\mathbf{X}},\\
\theta^{(4)}_{\mathbf{ik}} &= \Big|\sum_{\mathbf{j}=2}^{\mathbf{n}_i}\epsilon_{\mathbf{ij}}\phi_{\mathbf{k}}(s_{\mathbf{ij}})(s_{\mathbf{ij}} - s_{\mathbf{i},\mathbf{j}-1})\Big|, & Z^{(4)}_{\mathbf{k}} &\equiv 1,\\
\theta^{(5)}_{\mathbf{ik}} &= \sum_{\mathbf{j}=2}^{\mathbf{n}_i}|\epsilon_{\mathbf{ij}}|(s_{\mathbf{ij}} - s_{\mathbf{i},\mathbf{j}-1}), & k &
\end{aligned}
$$

Lemma 2 *For $\theta_{ik}^{(\ell)}$, $Z_k^{(\ell)}$, $\vartheta_{im}^{(\ell)}$ and $Q_m^{(\ell)}$ as defined in [illegible] and [illegible]*

$$|\xi_{ik}^{I} - \xi_{ik}| \le \sum_{\ell=1}^{5} \theta_{ik}^{(\ell)} Z_k^{(\ell)}, \qquad |\zeta_{im}^{I} - \zeta_{im}| \le \sum_{\ell=1}^{5} \vartheta_{im}^{(\ell)} Q_m^{(\ell)}. \tag{9}$$

The proof is in Section [illegible] in the [illegible] supplementary [illegible] (Supplement Part I) in the [illegible] C[illegible] rates of ξ_{ik}^{I} and ζ_{im}^{I}.

[illegible] that the sequences of [illegible] and [illegible] h_k and h_{mk} are employed to obtain the rates of f_k and f_{mk} for the regression functions f_k and f_{mk}, and that the density of ξ_k is denoted by p_k. Define

$$\theta_k(x) = [illegible]\, p_k(x)\{\frac{\pi_k^{X}}{\sqrt{n}h_X^2} + \frac{[illegible]}{\sqrt{n}b_X} + \sqrt{[illegible]_X^*}\},$$
$$\vartheta_{mk}(x) = [illegible]\, p_k(x)\{\frac{\pi_m^{Y}}{\sqrt{n}h_Y^2} + \frac{[illegible]}{\sqrt{n}b_Y} + \sqrt{[illegible]_Y^*}\}. \tag{10}$$

The [illegible] convergence rates θ_k and ϑ_{mk} of the regression functions to $f_k(x)$ and $f_{mk}(x)$ (see Theorem [illegible]) are as follows:

$$\theta_k(x) = \frac{\theta_k(x)}{h_k} + [illegible]|f_k''(x)|h_k^2 + \sqrt{\frac{[illegible] Y|x)\|K_1\|^2}{p_k(x)nh_k}}, \tag{11}$$
$$\vartheta_{mk}(x) = \frac{\theta_k(x)}{h_{mk}} + \vartheta_{mk}(x) + [illegible]|f_{mk}''(x)|h_{mk}^2 + \sqrt{\frac{[illegible]\ \zeta_m|x)\|K_1\|^2}{p_k(x)nh_{mk}}}.$$

Considering the predictions $\widehat{E}(Y|X)$ for the [illegible] response case and $\widehat{E}\{Y(t)|X\}$ for the functional response case, the numbers of eigenfunctions K and M used for approximation in the infinite dimensional processes X and Y generally tend to infinity as the sample size n increases. We here assume $K \le K_0$ and $M \le M_0$ in A[illegible] [illegible] since [illegible] follows from [illegible] that $K_0 \to \infty$, [illegible] on [illegible] the [illegible] eigenvalues λ_j are of [illegible] polynomial [illegible], and similarly for M_0. [illegible] notations [illegible] on [illegible] [illegible] the set of positive integers by $\mathcal{N}$ and $\mathcal{N}_k = \{1, \ldots, k\}$. Convergence rates θ_n^* and ϑ_n^* for the predictions [illegible] and e s fo o s

$$\theta_{\mathbf{n}}^{*} \quad \sum_{\mathbf{k}=1}^{\mathbf{K}}\Big\{\frac{\theta_{\mathbf{k}}\ \xi_{\mathbf{k}}}{h_{\mathbf{k}}} \quad -|f_{\mathbf{k}}''\ \xi_{\mathbf{k}}|h_{\mathbf{k}}^{2} \quad \sqrt{\frac{Y|\xi_{\mathbf{k}}|\|K_1\|^2}{p_{\mathbf{k}}\ \xi_{\mathbf{k}}\ nh_{\mathbf{k}}}}\Big\} \quad \Big|\sum_{\mathbf{k}\geq\mathbf{K}+1} f_{\mathbf{k}}\ \xi_{\mathbf{k}}\Big|,$$

$$\vartheta_{\mathbf{n}}^{*} \quad \sum_{\mathbf{k}=1}^{\mathbf{K}}\sum_{\mathbf{m}=1}^{\mathbf{M}}\Big\{\frac{\theta_{\mathbf{k}}\ \xi_{\mathbf{k}}}{h_{\mathbf{mk}}} \quad \vartheta_{\mathbf{mk}}\ \xi_{\mathbf{k}}\ |\psi_{\mathbf{m}}\ t| \quad -|f_{\mathbf{mk}}''\ \xi_{\mathbf{k}}|\psi_{\mathbf{m}}\ t|h_{\mathbf{mk}}^{2} \quad \sqrt{\frac{\zeta_{\mathbf{m}}|\xi_{\mathbf{k}}|\|K_1\|^2}{p_{\mathbf{k}}\ \xi_{\mathbf{k}}\ nh_{\mathbf{k}}}}|\psi_{\mathbf{m}}\ t|$$

$$\frac{\pi_{\mathbf{m}}^{\mathbf{y}}|f_{\mathbf{mk}}\ \xi_{\mathbf{k}}|}{}$$

No n ξ_{ik} η_{ik} τ_{ik}, one find s

$$|\xi_{ik} - \xi_{ik}| \le \{|\eta_{ik} - \eta_{ik}| \quad |\eta_{ik} - \xi_{ik}| \quad |\tau_{ik}|\}.$$

o o ss of ene y, ss e $\|\phi_k\|_\infty \ge$, $\|\phi'_k\|_\infty \ge$, $\|X_i\|_\infty \ge$ and $\|X'_i\|_\infty \ge$ o $\theta_{ik}^{(\ell)}$ and $Z_k^{(\ell)}$, ℓ , …, , e s e on e s of s nded y

$$\{\sum_{j=2}^{n_i} |X_i\, s_{ij} - \mu\, s_{ij}| \cdot |\phi_k\, s_{ij} - \phi_k\, s_{ij}| \quad |\mu\, s_{ij} - \mu\, s_{ij}| \cdot |\phi_k\, s_{ij}|\; s_{ij} - s_{i,j-1}\}$$

$$\le \{\sum_{j=1}^{n_i} |X_i\, s_{ij}| \quad |\mu\, s_{ij}| \quad {}^2\; s_{ij} - s_{i,j-1}\}^{1/2}\{\sum_{j=2}^{n_i} \phi_k\, s_{ij} - \phi_k\, s_{ij}\; {}^2\; s_{ij} - s_{i,j-1}\}^{1/2}$$

$$\{\sum_{j=1}^{n_i} \mu\, s_{ij} - \mu\, s_{ij}\; {}^2\; s_{ij} - s_{i,j-1}\}^{1/2}\{\sum_{j=2}^{n_i} \phi_k^2\, s_{ij}\; s_{ij} - s_{i,j-1}\}^{1/2}$$

$$\le \theta_{ik}^{(1)} Z_k^{(1)} \quad \theta_{ik}^{(2)} Z_k^{(2)}.$$

e second e on e s of s e ppe ond

$$|\eta_{ij} - \xi_{ik}| \le \|\, X_i \quad \mu'\phi_k \quad X_i \quad \mu\,\phi'_k\|_\infty \quad {}^{*}_{X} \le \theta_{ik}^{(3)} Z_k^{(3)}.$$

o e o e, e d e on e s of s nded y $\theta_{ik}^{(4)} Z_k^{(4)}$ $\theta_{ik}^{(5)} Z_k^{(5)}$

□

Proof of Theore o s p c y, deno e $\sum_{i=1}^{n}$ y $\sum_i$, w_i $K_1\{x - \xi_{ik}/h_k\}/nh_k$, w_i $K_1\{x - \xi_{ik}/h_k\}/nh_k$, and e θ_k $\theta_k\, x$ o , e o ne e s o $f_k\, x$ of e e e ss on nc on $f_k\, x$ an e e p c y en s

$$f_k\, t \quad \frac{\sum_i w_i Y_i}{\sum_i w_i} - \frac{\sum_i w_i\, \xi_{ik} - x}{\sum_i w_i} f'_k\, x,$$

e e

$$f'_k\, x \quad \frac{\sum_i w_i\, \xi_{ik} - x\, Y_i - \{\sum_i w_i\, \xi_{ik} - x \sum_i w_i Y_i\}/\sum_i w_i}{\sum_i w_i\, \xi_{ik} - x^2 - \{\sum_i w_i\, \xi_{ik} - x\}^2/\sum_i w_i}.$$

Le $f_k\, x$ e ypo e e s o , o ned y s s n e e s w_i and ξ_{ik} fo w_i, ξ_{ik} n and o e e $|f_k\, x - f_k\, x|$, one s o n fy e o de s of e d e ence s

$$D_1 \quad \sum_i w_i - \hat{w}_i, \quad D_2 \quad \sum_i w_i - \hat{w}_i Y_i,$$

$$D_3 \quad \sum_i w_i \xi_{ik} - w_i \xi_{ik}, \quad D_4 \quad \sum_i w_i \xi_{ik}^2 - w_i \xi_{ik}^2.$$

Con s de n D_1, o o ss of ene y, ss e e co pac s ppo of K_1 s , nce K_1 s L p s z con n o s on s s ppo ,

$$D_1 \leq \frac{c}{nh_k^2} \sum_i |\xi_{ik} - \xi_{ik}|\{I |x - \xi_{ik}| \leq h_k \quad I |x - \xi_{ik}| \leq h_k\},$$

fo s o e $c >$, e e I s an nd a o nc on Le a p e s fo e s e on e s of

$$\frac{1}{nh_k^2} \sum_i |\xi_{ik} - \xi_{ik}| I |x - \xi_{ik}| \leq h_k \leq \sum_{\ell=1}^{5} Z_k^{(\ell)} \frac{1}{nh_k^2} \sum_i \theta_{ik}^{(\ell)} I |x - \xi_{ik}| \leq h_k.$$

Apply n e cen eo e fo a ndo e of s and s B n s ey, 199 , p a e , o s n $\sum_i I |x - \xi_{ik}| \leq h_k / nh_k \xrightarrow{p} p_k\, x$, one nd s

$$\frac{1}{nh_k} \sum_i \theta_{ik}^{(\ell)} I |x - \xi_{ik}| \leq h_k \xrightarrow{p} p_k\, x\, E\, \theta_{ik}^{(\ell)},$$

p o ded a $E\, \theta_{ik}^{(\ell)} < \infty$ fo ℓ 1, . . . , No e a $E\theta_{ik}^{(1)} < \infty$, $E\theta_{ik}^{(3)} < \infty$ y A , $E\theta_{ik}^{(4)} \leq \sigma_X \sqrt{\ \ {}^*_X}$ and $E\theta_{ik}^{(5)} \leq |\mathcal{S}|\sigma_X$ y e C a y a z ne a y en

$$Z_k^{(1)} \frac{1}{nh_k^2} \sum_i \theta_{ik}^{(1)} I |x - \xi_{ik}| \leq h_k \quad O_p\{\frac{\pi_k^X}{\sqrt{n}h_X^2 h_k} p_k\, x\},$$

$$Z_k^{(2)} \frac{1}{nh_k^2} \sum_i \theta_{ik}^{(2)} I |x - \xi_{ik}| \leq h_k \quad O_p\{\frac{1}{\sqrt{n}b_X h_k} p_k\, x\},$$

$$Z_k^{(3)} \frac{1}{nh_k^2} \sum_i \theta_{ik}^{(3)} I |x - \xi_{ik}| \leq h_k \quad O_p\{\frac{\|\phi_k\|_\infty \ \ {}^*_X}{h_k} p_k\, x\},$$

$$Z_k^{(4)} \frac{1}{nh_k^2} \sum_i \theta_{ik}^{(4)} I |x - \xi_{ik}| \leq h_k \quad O_p\{\frac{\sqrt{\ \ {}^*_X}}{h_k} p_k\, x\},$$

$$Z_k^{(5)} \frac{1}{nh_k^2} \sum_i \theta_{ik}^{(5)} I |x - \xi_{ik}| \leq h_k \quad O_p\{\frac{\pi_k^X}{\sqrt{n}h_X^2 h_k} p_k\, x\}.$$

e no o n $(nh_k^2)^{-1}\sum_i |\xi_{ik} - \xi_{ik}|I(|x - \xi_{ik}| \le h_k) = O_p(\theta_k h_k^{-1})$ e y p o c e of e econd e n e de ed o y, o n

$$\frac{1}{nh_k}\sum_i I(|x-\xi_{ik}| \le h_k) \le \frac{1}{nh_k}\sum_i \{I(|x-\xi_{ik}| \le h_k) + I(\sum_{\ell=1}^{5}\theta_{ik}^{(\ell)} Z_k^{(\ell)} > h_k)\} \xrightarrow{p} p_k(x),$$

d n o $(nh_k^2)^{-1}\sum_i |\xi_{ik} - \xi_{ik}|I(|x - \xi_{ik}| \le h_k) = O_p(\theta_k h_k^{-1})$ en $D_1 = O_p(\theta_k h_k^{-1})$ fo o

An o y, one o $D_2 = O_p(\theta_k h_k^{-1})$, pp y n e C y z ne y fo $\theta_{ik}^{(\ell)}$, $\ell = 1, \ldots, 5$, nd o n e ndependence e een Y_i nd $\theta_{ik}^{(\ell)}$ fo $\ell = 1, \ldots, 5$, en e o en cond on A o D_3, o e

$$D_3 = \sum_i \{(w_i - \hat{w}_i)\xi_{ik} + (w_i - \hat{w}_i)(\xi_{ik} - \xi_{ik}) + w_i(\xi_{ik} - \xi_{ik})\} \equiv D_{31} + D_{32} + D_{33}.$$

en $D_{31} = O_p(\theta_k h_k^{-1})$, o y o D_1 e y o ee $D_{32} = o_p(D_{31})$

nce $D_{33} \le c\sum_{\ell=1}^{5} Z_k^{(\ell)}(nh_k^2)^{-1}\sum_i \theta_{ik}^{(\ell)} I(|x - \xi_{ik}| \le h_k)$ fo o e $c > 0$, one o $D_{33} = o_p(D_{31})$ e n $D_3 = O_p(\theta_k h_k^{-1})$ O n $|\xi_{ik}^2 - \xi_{ik}^2| \le |\xi_{ik} - \xi_{ik}| \cdot |\xi_{ik}| + (\xi_{ik} - \xi_{ik})^2$, one n o $D_4 = O_p(\theta_k h_k^{-1})$, n en fo D_3, nd $E\xi_{ik}^4 < \infty$ fo A Co n n e e fo D_ℓ, $\ell = 1, \ldots, 4$, nd pp y n y eo e d o $|f_k(x) - f_k(x)| = O_p(\theta_k h_k^{-1})$ n A, nd pp y n nd d y p o c e fo e ypo e ne oo e $f_k(x)$ co p e e e p oof of

o de e , dd on y one on y need o con de $\sum_i w_i\zeta_{im} - w_i\zeta_{im} = \sum_i \{(w_i - \hat{w}_i)\zeta_{im} + (w_i - \hat{w}_i)(\zeta_{im} - \zeta_{im}) + w_i(\zeta_{im} - \zeta_{im})\}$, e e e d e y ed n e e of o de $O_p(\vartheta_{mk})$ y o n

$$|\sum_i w_i(\zeta_{im} - \zeta_{im})| \le \sum_{\ell=1}^{5} Q_m^{(\ell)} \sum_i w_i \vartheta_{im}^{(\ell)} \le \frac{1}{nh_{mk}}\sum_{\ell=1}^{5} Q_m^{(\ell)}\sum_i \vartheta_{im}^{(\ell)} I(|x - \xi_{ik}| \le h_{mk})$$

en o e co p e e de on □

Proof of Theore n A, e de on of θ_n^* n o d, fo o n e o e en s o o n , no e

$$
\begin{aligned}
&\widehat{E}\{Y(t)|X\} - E\{Y(t)|X\} \\
\le\ & \sum_{\mathbf{k}=1}^{\mathbf{K}} \sum_{\mathbf{m}=1}^{\mathbf{M}} |f_{\mathbf{mk}}(\xi_{\mathbf{k}})\psi_{\mathbf{m}}(t) - f_{\mathbf{mk}}(\xi_{\mathbf{k}})\psi_{\mathbf{m}}(t)| + |\sum_{\mathbf{k}\geq \mathbf{K}+1} \sum_{\mathbf{m}\geq \mathbf{M}+1} f_{\mathbf{mk}}(\xi_{\mathbf{k}})\psi_{\mathbf{m}}(t)| \\
\le\ & \sum_{\mathbf{k}=1}^{\mathbf{K}} \sum_{\mathbf{m}=1}^{\mathbf{M}} |f_{\mathbf{mk}}(\xi_{\mathbf{k}}) - f_{\mathbf{mk}}(\xi_{\mathbf{k}})|\{|\psi_{\mathbf{m}}(t)| + |\psi_{\mathbf{m}}(t) - \psi_{\mathbf{m}}(t)|\} + |f_{\mathbf{mk}}(\xi_{\mathbf{k}})| \cdot |\psi_{\mathbf{m}}(t) - \psi_{\mathbf{m}}(t)| \\
& + |\sum_{(\mathbf{k},\mathbf{m})\in\mathcal{N}^2\setminus\mathcal{N}_K\times\mathcal{N}_M} f_{\mathbf{mk}}(\xi_{\mathbf{k}})\psi_{\mathbf{m}}(t)|.
\end{aligned}
$$

s p e s e con e ence e $\vartheta_{\mathbf{n}}^*$ n