

Weak universality results for singular SPDEs

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KPZ equation

The 1+1 dimensions KPZ equation is given by

$$\partial_t h = \partial_x^2 h + \mu(\partial_x h)^2 + \xi.$$

- $(t, x) \in \mathbf{R}^+ \times \mathbf{T}$.
- ξ is the space-time white noise.
- μ describes the strength of asymmetry.

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Renormalization: $\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \mu(\partial_x h_\varepsilon)^2 + \xi_\varepsilon - C_\varepsilon$.

- $C_\varepsilon = \frac{c}{\varepsilon} + \mathcal{O}(1)$.

Theorem(Hairer 13' 14') h_ε converges in probability in some sense as $\varepsilon \rightarrow 0$ independent of regularization.

- The solution is defined as the limit, denote the solution family by $\text{KPZ}(\mu)$.

Weak universality for KPZ equation

Microscopic model: $\partial_t \tilde{h} = \partial_x^2 \tilde{h} + \sqrt{\varepsilon} F(\partial_x \tilde{h}) + \tilde{\xi}$.

- $\sqrt{\varepsilon}$: weak asymmetry.
- F : even function.
- $\tilde{\xi}$: smooth space-time Gaussian field.

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Rescaled process $h_\varepsilon := \sqrt{\varepsilon} \tilde{h}(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) - C_\varepsilon t$ solves

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Theorem (Hairer-Quastel 18') If F is an **even polynomial**, then there exists $C_\varepsilon \rightarrow +\infty$ such that $h_\varepsilon \rightarrow KPZ(\mu)$.

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Theorem(Gubinelli-Perkowski 16', Yang 23'): If F is Lipschitz, then Hairer-Quastel universality holds **in law** at stationarity/non-stationarity.

- Require knowledge of **invariant measure**.
- The **microscopic** process should be **Markovian**.

Weak universality for KPZ equation

Heuristic explanation: Let Z_ε solves $\partial_t Z_\varepsilon = \partial_x^2 Z_\varepsilon + \xi_\varepsilon$ and $\Psi_\varepsilon = \partial_x Z_\varepsilon$. Then $u_\varepsilon := h_\varepsilon - Z_\varepsilon$ solves

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + \varepsilon^{-1} F(\sqrt{\varepsilon} \Psi_\varepsilon + \sqrt{\varepsilon} \partial_x u_\varepsilon) - C_\varepsilon.$$

- $\sqrt{\varepsilon} \Psi_\varepsilon$ is asymptotically distributed as $\mathcal{N}(0, \sigma^2)$.
- $\partial_x u_\varepsilon$ has order $\varepsilon^{-\kappa}$.

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Taylor expanding $\varepsilon^{-1} F(\sqrt{\varepsilon} \Psi_\varepsilon + \sqrt{\varepsilon} \partial_x u_\varepsilon)$:

$$\underbrace{\varepsilon^{-1} F(\sqrt{\varepsilon} \Psi_\varepsilon) - C_\varepsilon}_{\mu \Psi^\diamond} + \underbrace{\varepsilon^{-\frac{1}{2}} F'(\sqrt{\varepsilon} \Psi_\varepsilon)}_{2\mu \Psi} \partial_x u_\varepsilon + \frac{1}{2} \underbrace{F''(\sqrt{\varepsilon} \Psi_\varepsilon)}_{2\mu} (\partial_x u_\varepsilon)^2 + \mathcal{O}(\varepsilon^\kappa).$$

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Question: Does the weak universality result of Hairer and Quastel hold for $F \in \mathcal{C}^{2+}$?

Theorem(Hairer-Xu 19'): For KPZ equation, if even function $F \in \mathcal{C}^{7+}$ with polynomial growth, then there exist $C_\varepsilon \rightarrow +\infty$ and μ such that $h_\varepsilon \rightarrow KPZ(\mu)$.

regularity structure + convergence of stochastic terms

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Theorem(Kong-Z. 22+) The same result holds for even function $F \in \mathcal{C}^{2+}$ with polynomial growth.

Weak universality for Φ_3^4 equation

Rescaled process ϕ_ε solves

$$\partial_t \phi_\varepsilon = \Delta \phi_\varepsilon - \varepsilon^{-\frac{3}{2}} V'(\sqrt{\varepsilon} \phi_\varepsilon) + \xi_\varepsilon + C_\varepsilon \phi_\varepsilon.$$

Take $u_\varepsilon := \phi_\varepsilon - \Psi_\varepsilon$ where Ψ_ε solves $\partial_t \Psi_\varepsilon = \Delta \Psi_\varepsilon + \xi_\varepsilon$. Then u_ε solves

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Question: How about arbitrary $V \in \mathcal{C}^{4+}$?

Theorem(Furlan-Gubinelli 19'): For Φ_3^4 equation, if even function $V \in \mathcal{C}^{10+}$ with exponential growth, then there exist $C_\varepsilon \rightarrow +\infty$ and μ such that $\phi_\varepsilon \rightarrow \Phi_3^4(\mu)$.

paracontrolled structure + convergence of stochastic terms

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Main bounds

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Hairer-Xu 19': $\left| \mathbf{E} \prod_{j=1}^{2n} (\sin(\theta X_j)) \right| \lesssim \theta^{2n} \mathbf{E} \prod_{j=1}^{2n} X_j = \mathbf{E} \prod_{j=1}^{2n} (\theta X_j)$.

Hairer-Xu 19':

$$\left| \mathbf{E} \prod_{j=1}^{2n} [\sin(\alpha X_j) \cdot \mathcal{T}_{(1)}(\cos(\beta Y_j))] \right| \lesssim (|\alpha| + |\beta|)^{6n} \mathbf{E} \prod_{j=1}^{2n} (X_j Y_j^{\otimes 2}). \quad (\star)$$

- Similar bounds hold for multi-frequency.
- We need (\star) to prove the convergence of stochastic objects like

$$\varepsilon^{-\frac{3}{2}} \int K(x-y) F'(X) \mathcal{T}_{(1)}(F(Y)) dy \quad \text{in } \mathcal{C}^{-\frac{1}{2}-\kappa}$$

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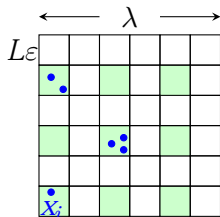
Xu 18': The bound with one frequency θ is independent of θ .

Proposition(Kong-Z. 22+) An **integral version** of (\star) holds with bound **independent** of α and β .

Solution

Final Solution:

$$\begin{aligned} & \varepsilon^{-\frac{3}{2}} \left\| \int \int K(x-y) \sin(\alpha X) \mathcal{T}_{(1)}(\cos(\beta Y)) \varphi^\lambda(x) \sum_{Q \in A} \mathbf{1}_{x \in Q} dx dy \right\|_{2n} \\ & \triangleq \varepsilon^{-\frac{3}{2}} \left\| \sum_{Q \in A} Z_Q \right\|_{2n} = \varepsilon^{-\frac{3}{2}} \left(\sum_{\sigma: [2n] \rightarrow A} \underbrace{\mathbf{E} Z_{\sigma(1)} \cdots Z_{\sigma(2n)}} \right)^{\frac{1}{2n}} \\ & = \int \int (\cdots) \mathbf{E} \prod_{i=1}^{2n} (\sin(\alpha X_i) \mathcal{T}_{(1)}(\cos \beta Y_i) \mathbf{1}_{x_i \in \sigma(i)}) d\vec{x} d\vec{y} \end{aligned}$$



Whether there exists singleton depends only on σ .

Bad σ is rare.

$$\mathcal{S} := \{ \sigma : \exists Q, |\sigma^{-1}(Q)| = 1 \}$$

$$|\mathcal{S}^c| \sim \left(\frac{\lambda}{\varepsilon}\right)^{nd}$$

Solution

Furthermore, by Holder inequality we get

$$\sum_{\sigma \notin \mathcal{S}} \mathbf{E} Z_{\sigma(1)} \cdots Z_{\sigma(2n)} \leq \sum_{\sigma \notin \mathcal{S}} \|Z_{\sigma(1)}\|_{2n} \cdots \|Z_{\sigma(2n)}\|_{2n}.$$

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We have a **uniform bound** on $\|Z_Q\|_{2n}$.

$$\begin{aligned} \|Z_Q\|_{2n} &= \left\| \int \int_{x \in Q} K(x-y) \sin(\alpha X) \mathcal{T}_{(1)}(\cos(\beta Y)) \varphi^\lambda(x) dx dy \right\|_{2n} \\ &\leq \int_{x \in Q} \varphi^\lambda(x) \left\| \int K(x-y) \sin(\alpha X) \mathcal{T}_{(1)}(\cos(\beta Y)) dy \right\|_{2n} dx \\ &\lesssim \varepsilon^{-\kappa} \int_{x \in Q} \varphi^\lambda(x) \left\| \int K(x-y) Y^{\diamond 2} dy \right\|_{2n} dx \\ &\lesssim \varepsilon^{-\kappa} \left(\frac{\varepsilon}{\lambda}\right)^d \varepsilon \end{aligned}$$

We would get

$$\begin{aligned} & \left(\sum_{\sigma \in \mathcal{S}^c} \|Z_{\sigma(1)}\|_{2n} \cdots \|Z_{\sigma(2n)}\|_{2n} \right)^{\frac{1}{2n}} \\ & \leq |\mathcal{S}^c|^{\frac{1}{2n}} \sup_Q \|Z_Q\|_{2n} \\ & \lesssim \left(\frac{\lambda}{\varepsilon}\right)^{d/2} \left(\frac{\varepsilon}{\lambda}\right)^d \varepsilon^{1-\kappa} \\ & = \left(\frac{\varepsilon}{\lambda}\right)^{d/2} \varepsilon^{1-\kappa} \\ & \lesssim \varepsilon^{\frac{3}{2}-\kappa} \lambda^{-\frac{1}{2}} \end{aligned}$$

- Weak universality results for **non-Gaussian** noise ξ .
Hairer-Shen 17': If F is even **polynomial**, then h_ε converges to KPZ for **non-Gaussian** $\tilde{\xi}$ (CLT).
Question: How about **general** F with **non-Gaussian** $\tilde{\xi}$?

Further questions

- Weak universality results for **non-Gaussian** noise ξ .
Hairer-Shen 17': If F is even **polynomial**, then h_ε converges to KPZ for **non-Gaussian** $\tilde{\xi}$ (CLT).
Question: How about **general** F with **non-Gaussian** $\tilde{\xi}$?
- Weak universality results for $F(x) = |x|$ **without** the use of invariant measure.