
X Y $X \times \mathbf{E}^n$ $Y \times \mathbf{E}^n$

()

1.

2.

3.

4. Nil⁴, Sol⁴_{m,n}, Sol⁴₀ Sol⁴₁

5. 1-1 ($\mathbf{H}^2 \times \mathbf{H}^2$)

6. 1-1 ()

7.

8. 1-1 ()

9. $\mathbf{H}^2 \times \mathbf{H}^2$

Domination Relations in Four-dimensional Geometries

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Abstract: This article studies the virtual domination relation in four-dimensional geometries. This relation forms a partial order among the geometries, and can be shown in a domination diagram. The author draws and proves the diagram. Furthermore, a geometry X dominates another geometry Y if and only if $X \times \mathbb{E}^n$ dominates $Y \times \mathbb{E}^n$.

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1 Introduction

Assumption: All manifolds in this article are assumed to be closed and oriented, unless otherwise indicated.

The beginning of everything is the notion of domination:

Definition 1.1: Given two manifolds M_1 and M_2 of the same dimension, we say that M_1 dominates M_2 if there exists a map of non-zero degree from M_1 to M_2 .

Domination between manifolds has been extensively studied. [1], for example, gives a topological criterion concerning the domination between highly-connected even-dimensional manifolds. In dimension three, domination problems are in a much more active discussion. In the survey paper [18], Shicheng Wang summarized important problems on the domination between 3-manifolds. Geometric decomposition of 3-manifolds plays an important role, so do geometric manifolds. [13], [19] and [5], for example, studied non-zero degree maps between geometric 3-manifolds.

This article is concentrated on the domination between geometric 4-manifolds. Our target is the following domination relation between geometries:

Definition 1.2: Given two geometries X and Y of the same dimension. We say that X **virtually dominates** Y (abbreviated as: **dominates**), if for every X -manifold M there is a finite cover $M' \rightarrow M$ such that M' dominates some Y -manifold N . The domination relation is written as $X \rightarrow Y$.

It is easy to see that the domination relation is transitive: Let X, Y, Z be three geometries, if $X \rightarrow Y$ and $Y \rightarrow Z$, then $X \rightarrow Z$.

The domination relation between four-dimensional geometries is shown in the following diagram:

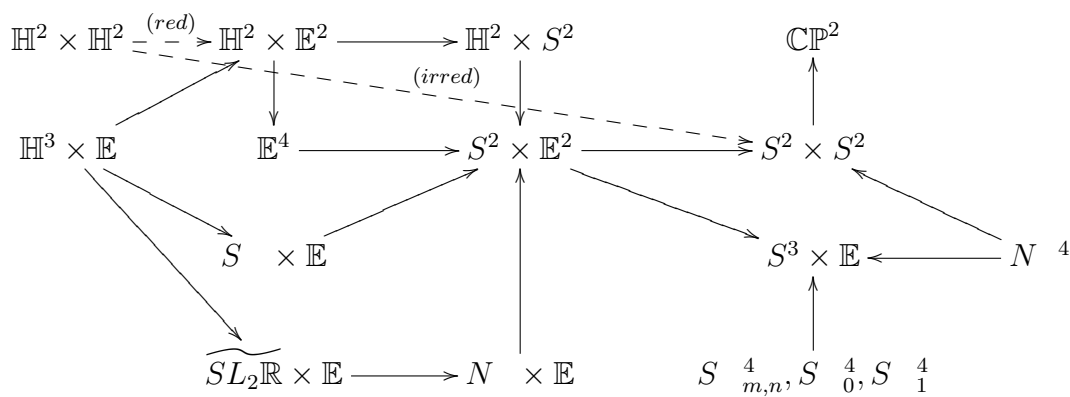


Diagram 1-1

Note that the arrow from $\mathbb{H}^2 \times \mathbb{H}^2$ separates into two parts: **reducible** (“red” in the diagram) and **irreducible** (“irred” in the diagram). Reducible $\mathbb{H}^2 \times \mathbb{H}^2$ -manifolds are finitely covered by a product of surfaces, while irreducible manifolds are not. This notion of irreducibility is essentially the same as the irreducibility of lattices in semisimple Lie groups.

The behavior of reducible and irreducible manifolds in domination are quite different. As we can see from the diagram, reducible manifolds (virtually) dominates $\mathbb{H}^2 \times \mathbb{E}^2$, but irreducible ones can only dominate $S^2 \times S^2$. What happens to irreducible manifolds is closely related to Margulis’s Normal Subgroup Theorem; see Section 9 for a detailed explanation.

There are 4 geometries that do not appear in Diagram 1-1: \mathbb{H}^4 , $\mathbb{C}\mathbb{H}^2$, F_4 and S^4 . All geometries dominate S^4 since all manifolds dominate S^4 . F_4 admits no compact quotient. We do not discuss \mathbb{H}^4 and $\mathbb{C}\mathbb{H}^2$ in this passage; we know little about them so far.

The domination diagram in dimension three is shown below. It is a part of the proof of Diagram 1-1.

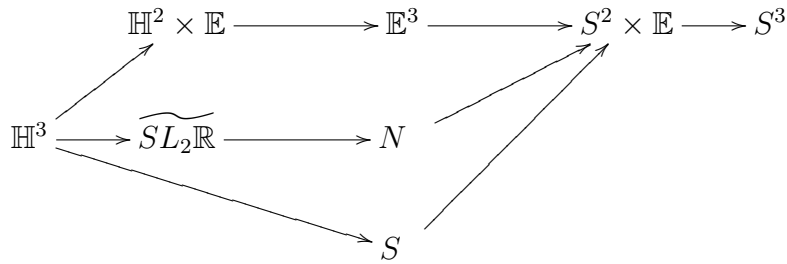


Diagram 1-2

Since there is geometric decomposition in dimension 3 as well as abundant topological tools, people are able to prove much stronger results for 3-manifolds. For example, we can prove that every 3-manifold 1-dominates only finitely many geometric 3-manifolds ([19, Corollary 1]). For another example, we can determine how many numbers of degrees (finitely or infinitely many) could there be in the maps between two 3-manifolds, according to their pieces in the geometric decomposition ([18, Theorem 1.3]).

However, there is no geometric decomposition for 4-manifolds, and there are not as much tools as in dimension three. It is generally hard for us to tell precisely whether a given 4-manifold dominates another one. But if we concentrate on geometric manifolds and virtual domination, we have good and rather complete results.

Insight into the domination diagram can also suggest us of interesting discoveries. Staring at Diagram 1-1 and 1-2, we can find that the map $X \mapsto X \times \mathbb{E}$ embeds Diagram 1-2 into Diagram 1-1 as a derived subgraph. Surprisingly, this holds in higher dimensions:

Corollary 7.5 (Embedding of Domination Diagram) *Suppose X, Y are two geometries of dimension (both) 2, 3 or 4. In addition, $X \neq \mathbb{H}^2 \times \mathbb{H}^2$ when Y is not contractible. Then $X \rightarrow Y$ if and only if $X \times \mathbb{E}^n \rightarrow Y \times \mathbb{E}^n$.*

This conclusion is mainly based on two theorems:

Proposition 7.1 (Product Geometry Splitting) *Let X be any geometry of dimension 2 or 3 or 4, n be any positive integer. Then every $X \times \mathbb{E}^n$ -manifold is finitely covered by $N \times T^n$ where N is an X -manifold.*

Proposition 7.3 (Domination Reduction) *Suppose M_1, M_2 are two manifolds of the same dimension. M_2 is a $K(G, 1)$ and $\pi_1(M_2)$ is torsion-free. If $M_1 \times S^1$ dominates $M_2 \times S^1$, then some finite cover of M_1 dominates M_2 .*

Proposition 7.1 is proved by investigating the geometries one by one. Proposition 7.3 is purely topological. We still do not know whether Proposition 7.1 generalizes to dimension > 4 , since its proof relies heavily on the properties of individual geometries.

Let us return to the proof of Diagram 1-1. In studying virtual domination, it is natural to develop the notion of Typical Covering Manifolds. The typical covering manifolds of a

geometry X is a "supreme set" of X -manifolds under covering. That is, every X -manifold is covered by one of them. Studying the domination between geometries is simplified to studying the domination between typical covering manifolds. Our two main criteria are direct consequence of definitions:

Proposition 5.1 *Let X and Y be geometries. If every typical covering manifold of X dominates some Y -manifold, then $X \rightarrow Y$ holds.*

Proposition 6.2 *Assume that X, Y are two geometries, and \mathcal{A} (resp. \mathcal{B}) is a set of typical covering manifold for X (resp. Y). If for every $M_1 \in \mathcal{A}$ and $M_2 \in \mathcal{B}$, M_1 does not dominate M_2 , then $X \not\rightarrow Y$.*

Finding the typical covering manifolds is thus a very important work. Fortunately, for almost all geometries we can find typical covering manifolds that are simple enough. For example, products of lower-dimensional manifolds. Table 3-1 contains our results. Most of the items in Table 3-1 are special cases of Proposition 7.1, and their proofs also generalizes to the proof of Proposition 7.1.

In proving Table 3-1, we frequently use the theorems on the discrete subgroup of Lie groups. By intersecting a lattice with appropriate closed subgroups, we may find sublattices and quotient lattices, and therefore construct fiber bundle structures. The bundle structure becomes simple enough (for example, becomes trivial) after taking suitable covers. This is a commonly used method in studying geometric manifolds. [15] and [16], for example, gives a detailed study of the fibration structures that occur in 4-dimensional geometries.

The remaining non-product geometries are $N^4, S^4_{m,n}, S^4_0$ and S^4_1 . Their geometric manifolds are mapping torus of T^3 or N^3 -manifold. This property is sufficient for us to determine their domination relation with other geometries.

The arrows in Diagram 1-1 will be easy conclusions of section 3 and 4. Proving non-arrows, however, needs more efforts. Some non-arrows are consequences of Diagram 1-2 and Corollary 7.5. Others have to be proved independently, mainly by using $K(G, 1)$. (As we know, $K(G, 1)$ is a bridge between topological information and group-theoretic information.)

The structure of this article is as follows. In section 2 we briefly introduce some basic notions, including the notion of geometry, Seifert bundles, monodromy and Euler numbers of torus bundles, and some discussion on Diagram 1-2. In section 3 we concern about Typical Covering Manifolds, and prove the important Table 3-1. In section 4 we study $N^4, S^4_{m,n}, S^4_0$ and S^4_1 -manifolds. In section 5, 6 and 8, we complete the proof of Diagram 1-1, except for $\mathbb{H}^2 \times \mathbb{H}^2$. In section 7, we prove the Embedding Theorem for domination relation. Finally in section 9, we discuss $\mathbb{H}^2 \times \mathbb{H}^2$ and complete the whole proof.

2 Preliminaries

1. Four-dimensional Geometries

A geometry is a pair (X, G, ρ) , where X is a connected simply-connected manifold, G is a connected Lie group, and ρ is an effective, transitive left action of G on X with

compact point stabilizer. We require two additional conditions:

① There exists a discrete subgroup Γ of G such that $\Gamma \backslash X$ is of finite volume.

② G is required to be maximal among those pairs (X, G', ρ') which satisfies all the conditions above.

We normally abbreviate a geometry (X, G, ρ) to X , and G is called the structure group of X . A manifold M is called to have the geometric structure of X , or to be of type X , or to be an X -manifold, if there is a discrete subgroup $\Gamma \subset G$ acting freely on X such that $M \cong \Gamma \backslash X$. Since X is simply-connected, we have $\pi_1(M) \cong \Gamma$.

Important thing: We require here that the structure group of a geometry be connected. For example, the structure group of S^1 is $I^0(S^1) = S^1$, which is a subgroup of index 8 in $I^1(S^1)$. In our sense, S^1 -manifolds are precisely torus bundle over the circle with Anosov glueing matrix (rather than being covered by them). Our requirement does not lose essential information, since the virtual domination allows passing to covering.

There are 8 three-dimensional geometries: $S^3, \mathbb{E}^3, \mathbb{H}^3, S^2 \times \mathbb{E}, \mathbb{H}^2 \times \mathbb{E}, N^3, \widetilde{SL_2\mathbb{R}}$ and S^1 . [12] gives a very good introduction to them, and we assume that the reader is familiar with their properties. The geometric structure on a 3-manifold is unique (if there is one), and according to geometric decomposition, every prime manifold can be cut by tori into geometric manifolds.

Four-dimensional geometries are classified by Filipkiewicz in [2]. The complete list is: $\mathbb{E}^4, S^4, \mathbb{H}^4, \mathbb{C}P^2, \mathbb{C}H^2, S^3 \times \mathbb{E}, \mathbb{H}^3 \times \mathbb{E}, N^4 \times \mathbb{E}, S^2 \times \mathbb{E}, \widetilde{SL_2\mathbb{R}} \times \mathbb{E}, S^2 \times S^2, \mathbb{E}^2 \times S^2, \mathbb{H}^2 \times S^2, \mathbb{H}^2 \times \mathbb{E}^2, \mathbb{H}^2 \times \mathbb{H}^2, S^4_0, F^4, N^4, S^4_{m,n}, S^4_1$. Their precise definitions can be found in the end of Filipkiewicz's article [2, "Summary of Maximal Geometries"].

It is natural to ask about the uniqueness of geometric structure in dimension four, and the answer is affirmative. In fact, the homotopy type completely determines the geometry, just as in dimension three:

Theorem 2.1 ([17, §10]) *Assume that M_1 is an X -manifold and M_2 is a X -manifold. If M_1 is homotopy equivalent to M_2 , then $X = X$. \square*

2. Seifert Bundles

The structure of Seifert bundle naturally occurs to geometric manifolds. Suppose that we have a geometry X with structure group G , and H is a normal subgroup of G . Let Γ be discrete in G and acts freely on X . Assume that the image Γ' of the projection map $\Gamma \rightarrow G/H$ is also discrete. In most cases, Γ is a lattice and $\Gamma \cap H$ is also a lattice (see section 3). There is a fibration sequence: $(\Gamma \cap H) \backslash X \rightarrow \Gamma \backslash X \rightarrow \Gamma' \backslash (X/H)$.

Normally Γ' does not act freely on X/H . Therefore, the fibration is an orbifold fibration. However, $\Gamma \backslash X$ is itself smooth. Hence we come to the following definition:

Definition 2.1 *A Seifert bundle is an orbifold fibration $M \rightarrow B$ in which M is smooth as orbifold.*

In dimension three, the notion of Seifert bundle is used to denote those which the general fiber is S^1 . Seifert invariants and Euler number can be defined on a Seifert bundle ([12, §3]). 3-dimensional Seifert bundles on geometric 2-orbifolds are all geometric, and the geometric type is classified by the Euler number and the geometry of the base orbifold ([12, theorem 5.3]).

Analogous to dimension three, some four-manifolds have the structure of Seifert bundles with torus fiber over 2-orbifolds. Seifert invariants and Euler numbers (which consists of two numbers) can be analogously defined. [15] and [16] gave a detailed introduction to 4-dimensional Seifert torus bundles and their geometry types. Note that in dimension four, not all Seifert torus bundle have geometric structure ([16, Theorem B]).

[14] gives an introduction to orbifolds and orbifold fibrations. Seifert bundles behave well with respect to pullback. When the base manifold is smooth, a Seifert bundle is an ordinary fiber bundle.

3. Monodromy and Euler Numbers of Torus Bundles

Given a torus bundle $T^n \rightarrow M \rightarrow B$, we have an exact sequence of fundamental group: $0 \rightarrow \pi_1(T^n) \rightarrow \pi_1(M) \rightarrow \pi_1(B) \rightarrow 1$. Conjugation induces an action of $\pi_1(B)$ on $\pi_1(T^n)$. If for every $\gamma \in \pi_1(B)$, the action of γ on $\pi_1(T^n)$ has determinant 1, then we say that M is an oriented torus bundle.

Remark: Since our definition of geometry requires that the structure group be connected (only contains orientation-preserving ones), all torus bundles arising in geometric manifolds are orientable. The reader can check case by case when we deal with respective geometries in section 3 and 4. In section 3, we can always take a double cover to make the bundle orientable, so this is not a serious problem. For these considerations, in this article we implicitly assume that all the torus bundles we meet are orientable.

Choose a basis of $\pi_1(T^n)$, and we have a homomorphism $\pi_1(B) \rightarrow SL_n(\mathbb{Z})$. This homomorphism is called **Monodromy**.

When $B = S^1$, monodromy has another name: **glueing matrix**. A T^n -bundle over S^1 can be regarded as a quotient of $T^n \times [0, 1]$ under an equivalence relation $(x, 0) \sim (Ax, 1)$. The matrix $A \in SL_n(\mathbb{Z})$ coincides with the monodromy of M .

When B is a surface, we can define the notion of Euler numbers to M . We start with the presentation of $\pi_1(M)$: let $\pi_1(B, \gamma) = \langle \gamma_i, \gamma_j \mid \prod [\gamma_i, \gamma_j] = 1 \rangle$ be the standard presentation of $\pi_1(B)$, $(\lambda_1 \dots \lambda_n)$ be a basis of $\pi_1(T^2)$, and A_i, B_i be the monodromy matrices along γ_i, γ_j . Arbitrarily choose lifts $\tilde{\gamma}_i, \tilde{\gamma}_j \in \pi_1(M)$ that are projected to γ_i, γ_j . Then $\pi_1(M)$ has presentation

$$\pi_1(M) = \langle \lambda_i, \tilde{\gamma}_i, \tilde{\gamma}_j \mid [\lambda_i, \lambda_j] = 1, \tilde{\gamma}_i (\lambda_1 \dots \lambda_n) \tilde{\gamma}_i^{-1} = (A_i \lambda_1 \dots \lambda_n) A_i, \tilde{\gamma}_j (\lambda_1 \dots \lambda_n) \tilde{\gamma}_j^{-1} = (A_j \lambda_1 \dots \lambda_n) B_j, \prod [\tilde{\gamma}_i, \tilde{\gamma}_j] = \prod \lambda_i^{-e_i} \rangle$$

for some $e_1 \dots e_n \in \mathbb{Z}$. The numbers (e_i) are called the Euler numbers of M .

For circle bundles over surfaces (i.e. $n = 1$), the minus Euler number is called the χ -invariant, which is explained as the obstruction to the existence of global sections. M can be regarded as the unit circle bundle of a complex line bundle E . The Euler number of M is the same as the Chern class of E .

When $n > 1$, the Euler numbers is not a good "invariant". It depend on many things: the choice of the base point γ , the choice of the basis of $\pi_1(F_p)$, and the choice of the lifts of γ_i, γ_j . Changing the basis of $\pi_1(F_p)$ results in a contravariant change of the Euler numbers. Defining the Euler numbers needs a base point because of monodromy: there is no uniform way to define coordinate systems on every fiber. Changing the lifts of $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ (such as replacing $\tilde{\gamma}_i$ by $\tilde{\gamma}_i \lambda_i$) results in a very complicated change of the Euler numbers.

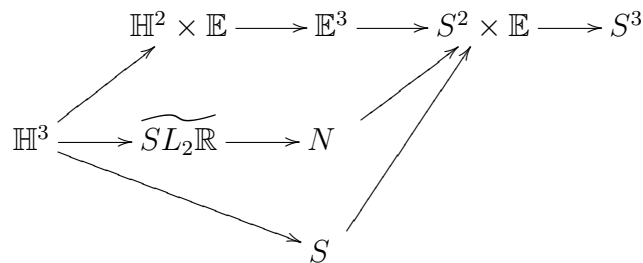
The Euler numbers, as elements in \mathbb{Z}^n , is subject to $SL_n(\mathbb{Z})$ -actions as we change the presentation of $\pi_1(M)$. However, whether the Euler numbers is 0 or not is a well-defined invariant, independent of the choice of all things.

When all monodromy matrices are trivial, things are much more simple. In this case, a torus bundle is just a $S^1 \times \dots \times S^1$ -bundle, and different factors does not affect each other. Thus we have separated Euler numbers with respect to the factors.

One important property (when the monodromy is trivial) is the splitting of M . If the Euler numbers have the form $(\dots, 0, \dots, 0)$, then M splits as $N \times T^{n-1}$. If we wish to rigorously explain this, then we can do as below. For simplicity, we use the case $n = 2$ as an example. Choose local trivializations of M : $M|_{U_i} \cong U_i \times T^2$. Since the monodromy is trivial, by choosing a particular basis on each trivialization, we can make the transition map split: $M|_{U_i} \cong U_i \times S^1 \times S^1$, and $\varphi_{ij} = \alpha_{ij} \times \beta_{ij}$. α_{ij} (resp. β_{ij}) is the transition map of some circle bundle N_1 (resp. N_2), and M is isomorphic to the pullback $N_1 \times_B N_2$. Suppose that the Euler numbers have the form $(\dots, 0)$, then N_2 is trivial, i.e. $N_2 = B \times S^1$. Therefore $M = N_1 \times S^1$.

4. Three-dimensional Domination Diagram

Recall Diagram 1-2



Proposition 2.2 *The arrows in Diagram 1-2 are indeed domination relations.*

Proof:

All geometries dominate S^3 since all manifolds dominate S^3 .

$S \rightarrow S^2 \times \mathbb{E}$ can be proved as follows: Let M be a S -manifold, then M is a T^2 -bundle over S^1 . We can find a section σ of M and a tubular neighborhood $U \supset \sigma$ such that the intersection of U with each fiber is homeomorphic to a disk. By contracting the complement of this disk on every fiber, we get a map from M to a S^2 -bundle over S^1 , which must be isomorphic to the trivial bundle $S^2 \times S^1$.

$\widetilde{SL_2\mathbb{R}} \rightarrow N$ can be proved as follows: A $\widetilde{SL_2\mathbb{R}}$ -manifold M is a Seifert bundle over a hyperbolic 2-orbifold B with nonzero Euler number. We first take a finite cover $\pi: B' \rightarrow B$ with B' smooth ([12, Theorem 2.5]). The pull-back bundle $M' = \pi^*M$ is then a circle bundle over a smooth hyperbolic surface, whose Euler number is an integral multiple of M . By doing vertical pinches ([10, Section 2]), we can produce a degree-one map $M' \rightarrow N$ where N is a circle bundle over T^2 with the same Euler number as M' (nonzero). Therefore N is a N -manifold. Using the same method, these arrows can be proved: $N \rightarrow S^2 \times \mathbb{E}$, and $\mathbb{H}^2 \times \mathbb{E} \rightarrow \mathbb{E}^3 \rightarrow S^2 \times \mathbb{E}$.

Finally, \mathbb{H}^3 dominates all other geometries ([6, Theorem 1.1]). \square

Proposition 2.3 *The non-arrows in Diagram 1-2 are not domination relations.*

We will leave this proposition to be proved in section 8; there our previous techniques can be used.

3 Typical Covering Manifolds

The typical covering manifolds for a geometry X is a set \mathcal{A} of X -manifolds, such that every X -manifold M has a finite cover $M' \rightarrow M$ such that $M' \in \mathcal{A}$. For example, a torus is a typical covering manifold for \mathbb{E}^4 by Biberbach Theorem ([9, Theorem 8.26]). Note that the set of typical covering manifolds is not unique.

Table 3-1 below lists examples of typical covering manifolds for some 4-dimensional geometries. In the lists and also in the remaining parts of this passage, we always assume that $g > 1$ in the expression " Σ_g ".

from now on, when talking about the geometries in Table 3-1, we make the convention that the typical covering manifolds for them are those described in the table.

	Geometry	Typical Covering Manifolds
(1)	\mathbb{E}^4	T^4
(2)	$\mathbb{H}^3 \times \mathbb{E}$	$(\mathbb{H}^3\text{-manifolds}) \times S^1$
(3)	$S^3 \times \mathbb{E}$	$S^3 \times S^1$
(4)	$S^2 \times \mathbb{E}^2$	$S^2 \times T^2$
(5)	$\mathbb{H}^2 \times \mathbb{E}^2$	$\Sigma_g \times T^2$ ($g > 1$)
(6)	$\mathbb{H}^2 \times S^2$	$\Sigma_g \times S^2$ ($g > 1$)
(7)	$\widetilde{SL_2\mathbb{R}} \times \mathbb{E}$	$(\widetilde{SL_2\mathbb{R}}\text{-manifolds}) \times S^1$
(8)	$N \times \mathbb{E}$	$(N\text{-manifolds}) \times S^1$
(9)	$S \times \mathbb{E}$	$(S\text{-manifolds}) \times S^1$
(10)	$S^2 \times S^2$	$S^2 \times S^2$

Table 3-1: typical covering manifolds for some geometries

The notion of typical covering manifolds works as follows. In order to prove $X \rightarrow$, we only need to prove that every typical covering X -manifold dominates some $-$ manifold. If we wish to prove $X \nrightarrow$, then we only need to prove that every typical covering X -manifold does not dominate any typical covering $-$ manifold. This idea leads to Proposition 5.1 and 6.2, which is our basic consideration for proving the diagram.

The following theorems on the discrete subgroups of Lie groups are very useful. A discrete subgroup Γ of a Lie group G is called a lattice if $\Gamma \backslash G$ has finite volume.

Lemma 3.1 ([9, Theorem 3.1]) *If G is solvable and $\Gamma \subset G$ is a lattice, then $\Gamma \backslash G$ is compact. \square*

Lemma 3.2 ([9, Theorem 3.3]) *Let G be solvable and N be its nilradical. If Γ is a lattice of G , then $\Gamma \cap N$ is a lattice of N , and $\Gamma / (\Gamma \cap N)$ is a lattice of G/N . \square*

Lemma 3.3 ([9, Corollary 2.3-1]) *Let G be nilpotent and G' be its commutator subgroup. If Γ is a lattice in G , then $\Gamma \cap G'$ is a lattice in G' . \square*

Lemma 3.4 ([9, Theorem 8.27]) *Let G be connected and N be the radical of G , such that G/N has no compact component. If Γ is a lattice in G , then $\Gamma \cap N$ is a lattice in N . \square*

Proof of Table 3-1:

In the following paragraphs, we fix our notation: $\Gamma \cong \pi_1(M)$ is a discrete subgroup of the structure group of X , with $M = \Gamma \backslash X$ compact. Denote by G the structure group of X . $\Gamma \backslash G$ is a principle bundle over M , and the fiber is the point stabilizer of X . Hence $\Gamma \backslash G$ is compact.

(1) $X = \mathbb{E}^4$:

This follows from Bieberbach Theorem ([9, Theorem 8.26]) which says that every compact flat manifold is covered by flat torus. \square

(2) $X = \mathbb{H}^3 \times \mathbb{E}$:

In this case, $\Gamma \subset I^{-1}(\mathbb{H}^3) \times I^{-1}(\mathbb{E}) =: G_1 \times \mathbb{R}$. From Lemma 3.4 we know that $\Gamma \cap \mathbb{R}$ is a lattice; assume that $\frac{1}{\ell} \in \mathbb{R}$ such that $\Gamma \cap \mathbb{R} = \mathbb{Z} \cdot \frac{1}{\ell}$. Let H be the image of the projection map $\Gamma \hookrightarrow G_1 \times \mathbb{R} \rightarrow G_1$; H is also discrete. There is an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \Gamma \rightarrow H \rightarrow 0$$

which expresses M as a Seifert bundle (with circle fiber) over the hyperbolic orbifold H/\mathbb{H}^3 . Every hyperbolic 3-orbifold has a finite smooth cover. To prove this, we need only look at the remark below ([12, Theorem 2.5]), and replace the group $PSL_2(\mathbb{R})$ by $PSL_2(\mathbb{C})$.

By taking pull-back, we have a finite cover $M' \rightarrow M$ such that M' is a circle bundle over a hyperbolic 3-manifold N . There is an exact sequence: $0 \rightarrow \mathbb{Z} \rightarrow \pi_1(M') \rightarrow \pi_1(N) \rightarrow 0$, where $\pi_1(N) \subset G_1$.

Let Σ be any immersed surface in N and let ι be the immersion map. Let $\{ \alpha_i, \beta_i \}$ be a set of standard generator of $\pi_1(\Sigma)$ and let $\alpha'_i = \iota_*(\alpha_i)$, $\beta'_i = \iota_*(\beta_i)$. Clearly $\prod [\alpha'_i, \beta'_i] = 1 \in \pi_1(N)$. Let $\bar{\alpha}_i, \bar{\beta}_i$ be any lift of α'_i, β'_i in $\pi_1(M')$; the Euler number $\langle \bar{\alpha}_i, \bar{\beta}_i \rangle$ satisfies the relation $\prod [\bar{\alpha}_i, \bar{\beta}_i] = \ell^{-e(j^*M')}$. But since $\pi_1(M') \subset G \times \mathbb{R}$, $\bar{\alpha}_i$ and $\bar{\beta}_i$ can be written as (α'_i, x_i) and (β'_i, y_i) , and we immediately have $\langle \bar{\alpha}_i, \bar{\beta}_i \rangle = 0$.

Let e be the Euler class of M' , i.e. the Euler class of the line bundle $E = M' \times_{S^1} \mathbb{C}$. By the above paragraph, e is mapped to zero under the map $H^2(N, \mathbb{Z}) \rightarrow H^2(N, \mathbb{C})$.

$$\begin{matrix} (M, E) \in \mathcal{C}H \\ \downarrow \\ (N, \gamma) \in \pi_1(M) \end{matrix}$$

(3) $X = S^2 \times \mathbb{E}^2$:

The structure group is $SO(3) \times I^{-1}(\mathbb{E}^2)$. $\Gamma \cap SO(3) = 1$ because Γ acts without fixed point on X . Thus the projection $\Gamma \rightarrow I^{-1}(\mathbb{E}^2)$ induces an isomorphism of Γ with its image. We can check that the image must be discrete and has compact quotient. This expresses M as a S^2 -bundle over the T^2 . All S^2 -bundles over T^2 are trivial.

the form $(0, 0, *)$. Note that we have already proved that $\Gamma \cap N$ is a lattice.

We have the following exact sequence that gives a torus bundle structure to M :

$$0 \rightarrow \mathbb{Z}^2 (= \Gamma \cap N) \rightarrow \Gamma \rightarrow \mathbb{Z}^2 (= \Gamma / (\Gamma \cap N)) \rightarrow 0$$

By definition, N is the center of N' , hence the monodromy is trivial. Choose the base $(,)$ of $\Gamma \cap N$ such that $\in (N)$. The Euler numbers now have the form $(*, 0)$, and this shows that M splits as a product of S^1 and a N -manifold. \square

The following statement is now trivial (but will be used later):

Proposition 3.1 *Let M be a typical covering manifold of $N \times \mathbb{E}$. Then M is a T^3 -bundle over S^1 . Furthermore, every finite cover of M is again a typical covering manifold.*

Remark: The glueing map for M is of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

(9) $X = S \times \mathbb{E}$:

The structure group of $S \times \mathbb{E}$ is $S \times \mathbb{R}$. $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ (where \mathbb{R} is the direct summand and $\mathbb{R}^2 \subset S$) is the nilradical of $S \times \mathbb{E}$. By Lemma 3.2, $\Gamma' = \Gamma \cap \mathbb{R}^3$ is a lattice, and the image of projection $\Gamma \hookrightarrow S \times \mathbb{R} \rightarrow (S \times \mathbb{R})/\mathbb{R}^3 \cong \mathbb{R}$ is equal to \mathbb{Z} for some $\in \mathbb{R}$. Assume that $\in S$ is any element that projects to $.$

For any $\sharp = (, \mathcal{Y},) \in \Gamma'$, $\sharp^{-1} - \sharp = ((, \mathcal{Y}) \left(\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} - I \right), 0)$. As the choice of \sharp is arbitrary, $\Gamma' \cap \mathbb{R}^2$ ($\mathbb{R}^2 \subset S$) must be a lattice in \mathbb{R}^2 . Therefore we can choose a set of basis $\sharp_1 = (\sharp_{11}, \sharp_{12}, 0)$, $\sharp_2 = (\sharp_{21}, \sharp_{22}, 0)$, $\sharp_3 = (\sharp_{31}, \sharp_{32}, \sharp_{33})$ of Γ' . Under the basis $\{\sharp_1, \sharp_2\}$, conjugation by \in is represented by an integral Anosov matrix C .

There exists $\sharp_1, \sharp_2 \in \mathbb{Z}$ such that $\in \sharp_3^{-1} = \sharp_3 + \sharp_1 \sharp_1 + \sharp_2 \sharp_2$ since the conjugation \in^{-1} preserves Γ' . From this formula we get

$$\begin{pmatrix} \sharp_{31} \\ \sharp_{32} \end{pmatrix} = \left(\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} - I \right)^{-1} (\sharp_1 \sharp_1 + \sharp_2 \sharp_2) = (\sharp_1 \ \sharp_2) (C - I)^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus there exists integer numbers \sharp_1, \sharp_2 such that $\sharp_4 = \in \sharp_1 + \sharp_2 + \sharp_3 = (0, 0, \sharp_4)$ lies in the \mathbb{R} summand of $S \times \mathbb{R}$. The subgroup Γ_1 generated by $\{\sharp_1, \sharp_2, \sharp_4, \in\}$ is an index subgroup of Γ . Furthermore, the conjugation by \in on $(\sharp_1, \sharp_2, \sharp_4)$ is represented by the matrix $A = \text{diag}(C', 1)$ where C' is an Anosov matrix. Now let M' be the covering space corresponding to Γ_1 , and we know that M' is a product of a S -manifold and S^1 . \square

Sometimes we need to know the structure of all $S \times \mathbb{E}$ -manifolds. Already we have:

Proposition 3.2 *Any $S \times \mathbb{E}$ -manifold has the structure of a T^3 -bundle over S^1 . In the basis of $(\sharp_1, \sharp_2, \sharp_3)$ described above, the glueing matrix is of the form:*

$$\begin{pmatrix} C & \\ 0 & 0 & 1 \end{pmatrix} \text{ (} C \text{ is Anosov)}$$

(10) $X = S^2 \times S^2$:

$I^{-1}(S^2 \times S^2) = SO(3) \times SO(3)$ is compact. \square

4 Manifolds of N^4, S^4, S^4_0 and S^4_1 Geometry

In this section, we will determine the structure of $N^4, S^4_{m,n}, S^4_0$ and S^4_1 -manifolds. One common point of these geometries is that their point stabilizers are trivial, i.e. they are the structure group of themselves ([2, Section 6.5]). Manifolds of these geometries are bundles over S^1 with T^3 or N^4 -manifold fiber.

1. N^4 -manifolds

By definition, $N^4 = \mathbb{R}^3 \rtimes_A \mathbb{R}$, in which

$$A(\cdot) = \begin{pmatrix} 1 & & 2/2 \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}$$

N^4 is nilpotent. Denote by x_1, x_2, x_3 ; the standard coordinates; the commutator subgroup is $N = [N^4, N^4] = \mathbb{R}(x_1, x_2)$. By lemma 3.3, if Γ is a lattice in N^4 , then $\Gamma \cap N$ is again a lattice, and the quotient $\Gamma' = \Gamma/\Gamma \cap N$ is a lattice in $N^4/N = \mathbb{R}^2$.

We can get more information about $\Gamma \cap N$. Suppose $v = (v_1, v_2, 0; 0) \in \Gamma \cap N$, then for any $w \in \Gamma'$, $w^{-1}v$ lies in \mathbb{R}^2 . There must exist a w such that $w^{-1}v \neq v$; hence $\Gamma \cap \mathbb{R}^2$ is nonzero. We can therefore assume that $\Gamma \cap N$ is generated by $w = (x_1, 0, 0; 0)$ and $v = (x_1, x_2, 0, 0)$.

Assume $w = (x_1, x_2, x_3; 4), y = (y_1, y_2, y_3; y_4)$ are two elements of Γ that project to a basis of Γ' (i.e. $3y_4 - 4y_3 \neq 0$). By concrete calculation we have the following relation:

$$(w \ y)^{-1} = (w \ y) \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ (for some } \lambda \in \mathbb{Z} \text{)}$$

The same relation holds for y . Changing the basis elements w, y we can assume that w commutes with y . This implies $x_4 = 0$. Finally we calculate $[y, w]$:

$$[y, w] = (2y_4 + 3y_4^2/2, 3y_4, 0; 0)$$

we find that $[w, y] = a \cdot b$ for $a, b \in \mathbb{Z}, b \neq 0$.

The exact sequence $0 \rightarrow \mathbb{Z}(w, y) \rightarrow \Gamma \rightarrow \mathbb{Z}y \rightarrow 0$ gives M the structure of T^3 -bundle over S^1 . To summarize, we have:

Proposition 4.1 *If M is a N^4 -manifold, then M is a T^3 -bundle over S^1 with glueing matrix*

$$A = \begin{pmatrix} 1 & \lambda & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}$$

such that $\lambda \neq 0, b \neq 0$. \square

Calculating the homology groups of M , we have:

Proposition 4.2 $H_1(M, \mathbb{Z}) = \mathbb{Z}^2 \oplus (\quad), H_2(M, \mathbb{Z}) = \mathbb{Z}^2 \oplus (\quad), H_3(M, \mathbb{Z}) = \mathbb{Z}^2$.

Proof: We use the following mapping torus long exact sequence; its proof can be found in [3, Section 2.2, Example 2.48].

$$\begin{aligned}
 H_3(T^3, \mathbb{Z}) \xrightarrow{0} H_3(T^3, \mathbb{Z}) \longrightarrow H_3(M, \mathbb{Z}) \longrightarrow H_2(T^3, \mathbb{Z}) \xrightarrow{1-\Lambda^2 A} H_2(T^3, \mathbb{Z}) \rightarrow \\
 \rightarrow H_2(M, \mathbb{Z}) \longrightarrow H_1(T^3, \mathbb{Z}) \xrightarrow{1-A} H_1(T^3, \mathbb{Z}) \longrightarrow H_1(M, \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow 0
 \end{aligned}$$

Let $\{y, z\}$ be a basis of $H_1(T^3, \mathbb{Z})$, such that A is represented by the matrix in Proposition 4.1.

We have $(1 - A) = \mathbb{Z} \oplus (\dots)$ where the \mathbb{Z} -summand is generated by \dots .

Under the basis $\{y \wedge z, z \wedge y, y \wedge x\}$ of $H_2(T^3, \mathbb{Z})$,

$$\Lambda^2 A = \begin{pmatrix} 1 & \lambda & \lambda & - \\ 0 & 1 & & \\ 0 & 0 & & 1 \end{pmatrix}$$

Hence $(1 - \Lambda^2 A) = \mathbb{Z} \oplus (\dots)$ where the \mathbb{Z} -summand is generated by $y \wedge z$. Furthermore we have $(1 - A) = \mathbb{Z}$ which is generated by \dots . Thus $H_2(M, \mathbb{Z}) = \mathbb{Z}^2 \oplus (\dots)$.

$(1 - \Lambda^2 A) = \mathbb{Z}$ is generated by $y \wedge z$, and hence $H_3(M, \mathbb{Z}) = \mathbb{Z}^2$. \square

Remark: From the proof of Proposition 4.1, we can find the generator of the torsion-free parts of the homology groups. $H_1(M, \mathbb{R})$ is generated by two elements: α , being represented by a section of the bundle, and β , a 1-dimensional "subspace" of the T^3 fiber spanned by \dots , which is actually a S^1 . $H_2(M, \mathbb{R})$ is generated by two elements: γ , a hyperplane in T^3 spanned by $\{y, z\}$, and δ , a subbundle of M — its fiber is a 1-dimensional subspace in T^3 spanned by \dots . $H_3(M, \mathbb{R})$ is generated by two elements: ϵ , the fiber T^3 , and ϕ , a subbundle of M whose fiber is the hyperplane in T^3 generated by $\{y, z\}$.

Using these generators Proposition 4.1 becomes

He n d R e

Proposition 4.3 $S^4_{m,n}$ -manifold has the structure of T^3 -bundle over S^1 with Anosov glueing matrix $A \in SL_3(\mathbb{Z})$, and A has three different real eigenvalues. \square

Proposition 4.4 Let M be a $S^4_{m,n}$ -manifold. Then $H_1(M, \mathbb{R}) = \mathbb{R}$ and $H_2(M, \mathbb{R}) = 0$.

Proof: Again we use the mapping torus sequence. The sequence has the same form as in Proposition 4.2, but the matrix A here is Anosov. Since $\Lambda^2 A = A^{-1}$, $\Lambda^2 A$ is also Anosov. Hence in real coefficient, $1 - A$ and $1 - \Lambda^2 A$ are both isomorphisms. The homology groups are now clear. \square

3. S^4_0 -manifolds

By definition, the structure group of S^4_0 is $G = \mathbb{R}^3 \rtimes_A (\mathbb{R} \times SO(2))$ where

$$A(t, \theta) = \begin{pmatrix} {}^t \cos \theta & {}^t \sin \theta & 0 \\ -{}^t \sin \theta & {}^t \cos \theta & 0 \\ 0 & 0 & -2t \end{pmatrix}$$

and the geometry is $S^4_0 = G/SO(2)$. Let $\Gamma \subset G$ be discrete and $M = \Gamma \backslash X$ is compact. $\Gamma \backslash G$ is a S^1 -bundle over X and hence is compact; therefore Γ is a lattice in G . G is solvable and its nilradical is $N = \mathbb{R}^3$. By Lemma 3.2, $\Gamma' = \Gamma \cap N \subset N$ and $H = \Gamma/\Gamma' \subset G/N = \mathbb{R} \times SO(2)$ are both lattices.

There are only two kinds of lattices in $\mathbb{R} \times SO(2)$: \mathbb{Z} and $\mathbb{Z} \oplus \mathbb{Z}_n$. If $H \cong \mathbb{Z}$, then we suppose that H is generated by (t_0, θ_0) , $t_0 \neq 0$. In this case, M is a T^3 -bundle over S^1 with glueing matrix $A \in SL_3(\mathbb{Z})$ which is conjugate to

$$\begin{pmatrix} {}^{t_0} \cos \theta_0 & {}^{t_0} \sin \theta_0 & 0 \\ -{}^{t_0} \sin \theta_0 & {}^{t_0} \cos \theta_0 & 0 \\ 0 & 0 & -2t_0 \end{pmatrix}$$

If $H \cong \mathbb{Z} \oplus \mathbb{Z}_n$, then suppose that H is generated by (t, θ) , $t \neq 0$, and $(0, \phi)$, where $\phi = 2\pi/n$. The conjugation action by (t, θ) and $(0, \phi)$ both preserves $\Gamma' \cong \mathbb{Z}^3$ and are isomorphisms on it. They are represented by matrices

$$A = \begin{pmatrix} {}^t \cos \theta & {}^t \sin \theta & 0 \\ -{}^t \sin \theta & {}^t \cos \theta & 0 \\ 0 & 0 & -2t \end{pmatrix}, B = \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) & 0 \\ \sin(2\pi/n) & \cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let (x, y, z) be standard coordinates in \mathbb{R}^3 . We claim that Γ' has non-empty intersection with the z -axis. Otherwise, there must exist $\mathbf{v} = (v_1, v_2, v_3) \in \Gamma'$ such that $(v_1, v_2) \neq (0, 0)$ and $v_3 \neq 0$. However, $\mathbf{v} + B\mathbf{v} + B^2\mathbf{v} + \dots + B^{n-1}\mathbf{v}$ is nonzero and must lie in the z -axis, which is a contradiction.

Clearly (t, θ) is an eigenvector of A with eigenvalue e^{-2t} . However, the z -axis has non-empty intersection with Γ' , which implies that e^{-2t} must be an integer, and since $e^{-2t} \neq 0$ it is not equal to 1. $\det(A)$, which is an integer, is divisible by e^{-2t} , hence cannot be 1. This leads to a contradiction, and hence $H \cong \mathbb{Z} \oplus \mathbb{Z}_n$ cannot happen.

Summarizing our results, we have:

Proposition 4.5 S^4_0 -manifold has the structure of T^3 -bundle over S^1 with Anosov glueing matrix $A \in SL_3(\mathbb{Z})$. A either has one real and two complex eigenvalues, or has three real eigenvalues in which two are equal. \square

For the converse statements, we have:

Proposition 4.6 *Let M be a T^3 -bundle over S^1 with glueing matrix A , and A is an Anosov matrix. Then M has $S^4_{m,n}$ or S^4_0 structure.*

Proof: A either has ①: three distinct real eigenvalues, or ②: a double eigenvalue, or ③: one real and two complex eigenvalues.

①: If A has three real eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3$, then suppose its characteristic polynomial is $\chi_A(\lambda) = \lambda^3 - a\lambda^2 + b\lambda - 1$, where we let the numbers a, b be the same as the numbers a, b in $S^4_{m,n}$. A is diagonalizable: $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ for some real matrix P . Let \mathbf{v}_i ($i = 1, 2, 3$) be the column vectors of P , and the lattice $\Gamma = \langle \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2 \rangle$, $\mathbf{v}_i \in \mathbb{Z} \in S^4_{m,n}$ has $\Gamma \backslash S^4_{m,n} = M$.

③ can be proved by the same method as ①. ② is just a special case of ③. \square

Proposition 4.7 *If M is a S^4_0 -manifold, then $H_1(M, \mathbb{R}) = \mathbb{R}$ and $H_2(M, \mathbb{R}) = 0$.*

Proof: it is the same as Proposition 4.4 \square

4. S^4_1 -manifolds

By definition, $S^4_1 = N \rtimes_A \mathbb{R}$, where

$$A(\lambda) : \begin{pmatrix} 1 & & \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & t & \\ 0 & 1 & -ty \\ 0 & 0 & 1 \end{pmatrix}$$

Note that the center of S^4_1 is spanned by $\mathbf{e}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. S^4_1 is solvable and its nilradical is N . If $\Gamma \subset S^4_1$ is a lattice, then by Lemma 3.2, $\Gamma' = \Gamma \cap N$ is also a lattice, and $\Gamma/\Gamma' = \mathbb{Z} \subset S^4_1/N = \mathbb{R}$. Let $L \in \Gamma$ be any element which projects to 1 .

The structure of Γ' is quite clear. The group N has an exact sequence $0 \rightarrow \mathbb{R} \rightarrow N \rightarrow \mathbb{R}^2 \rightarrow 0$, where the first term \mathbb{R} is spanned by \mathbf{e}_3 . There is also an exact sequence of Γ' which is compatible with that of N :

$$0 \rightarrow \Gamma' \cap \mathbb{R} (\cong \mathbb{Z}) \rightarrow \Gamma' \rightarrow \Gamma'/(\Gamma' \cap \mathbb{R}) (\cong \mathbb{Z}^2) \rightarrow 0$$

This expresses $E = \Gamma' \backslash N$ as a circle bundle over T^2 (with nonzero Euler number).

The conjugation action by L induces an isomorphism of Γ . Since $[L, L] = 1$, this isomorphism respects the exact sequence of Γ' :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma' \cap \mathbb{R} & \longrightarrow & \Gamma' & \longrightarrow & \Gamma'/(\Gamma' \cap \mathbb{R}) (\cong \mathbb{Z}^2) \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow c \\ 0 & \longrightarrow & \Gamma' \cap \mathbb{R} & \longrightarrow & \Gamma' & \longrightarrow & \Gamma'/(\Gamma' \cap \mathbb{R}) (\cong \mathbb{Z}^2) \longrightarrow 0 \end{array}$$

By looking at the definition of S^4_1 , we know that C is an Anosov matrix.

$M = \Gamma \backslash S^4_1$ is a E -bundle over S^1 with glueing map γ . By the above discussion, M can be described as:

$$\begin{array}{ccccc} S^1 & \longrightarrow & E & \longrightarrow & T^2 \\ \downarrow = & & \downarrow f & & \downarrow C \\ S^1 & \longrightarrow & E & \longrightarrow & T^2 \end{array}$$

i.e. F is a bundle isomorphism of E covering an Anosov map of T^2 . This describes the structure of S^4_1 -manifolds.

Summarizing our results:

Proposition 4.8 *Let M be a S^4_1 -manifold. Then there exists a 3-manifold E which is a circle bundle over T^2 (with nonzero Euler number), and a bundle isomorphism f of E , such that M is a E -bundle over S^1 with glueing map γ . \square*

The converse conclusion is also correct:

Proposition 4.9 *Let E be a circle bundle over T^2 with Euler number $e \neq 0$. M is a E -bundle over S^1 with glueing map $\gamma \in \text{Aut}(E)$, such that M is a bundle isomorphism of E which covers an Anosov map C of T^2 . Then M has S^4_1 geometry.*

Remark: The isotopy class of bundle isomorphism is classified by $[T^2, S^1] = H^1(T^2, \mathbb{Z})$, and is determined by the induced isomorphism γ_* on π_1 . What we essentially prove here is that all possible bundle isomorphism γ can be realized.

Proof: $\pi_1(M)$ is generated by $\pi_1(E)$ and an additional element K , which satisfies the relation $K^2 K^{-1} = \gamma_*(K)$ for $K \in \pi_1(E)$.

Let $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be standard generators of N .

For a real number λ , denote by x^λ the matrix $\begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (y^λ , z^λ is analogous).

We first embed $\pi_1(E)$ into N . Suppose that $\pi_1(E) = \Gamma' \subset N$ is generated by $\alpha_1 = x^\lambda y^\nu$, $\alpha_2 = y^\mu z^t$ and $\alpha_3 = z^k$. The real numbers λ, ν, μ, t, k are to be determined. The Euler number of E is $e (\neq 0)$ implies that $[\alpha_1, \alpha_2] = \alpha_3^k$, which is equivalent to $\lambda - \nu = k$.

Let $C = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$. Then we can suppose that $\gamma_*(\alpha_1) = \alpha_1^{p/r} \alpha_2^{b_1}$, $\gamma_*(\alpha_2) = \alpha_2^{q/s} \alpha_1^{b_2}$ and $\gamma_*(\alpha_3) = \alpha_3^k$. Since C is Anosov,

$$\begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix}^{-1} C \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix} = \begin{pmatrix} l & 0 \\ 0 & -l \end{pmatrix}$$

for some real numbers p', r', q', s', l . By multiplying with a constant number, we can make

$\Gamma = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$

$T^2 = \mathbb{R}^2 / \Gamma$

Proposition 4.10 *Let M be a S^1 -manifold. Then $H_1(M, \mathbb{R}) = \mathbb{R}$, $H_2(M, \mathbb{R}) = 0$.*

Proof: The notations E , has the same meaning as above.

The homology of E can be calculated by the Wang sequence:

$$0 \longrightarrow H^1(T^2, \mathbb{R}) \longrightarrow H^1(E, \mathbb{R}) \longrightarrow H^1(S^1, \mathbb{R}) \xrightarrow{c} H^2(T^2, \mathbb{R})$$

which implies that $H_1(E, \mathbb{R}) = H_2(E, \mathbb{R}) = \mathbb{R}^2$.

Then we apply the mapping torus sequence:

$$\begin{aligned} H_3(E, \mathbb{R}) \xrightarrow{0} H_3(E, \mathbb{R}) \longrightarrow H_3(M, \mathbb{R}) \longrightarrow H_2(E, \mathbb{R}) \xrightarrow{1-f^*|_{H_2}} H_2(E, \mathbb{R}) \rightarrow \\ \rightarrow H_2(M, \mathbb{R}) \longrightarrow H_1(E, \mathbb{R}) \xrightarrow{1-f^*|_{H_1}} H_1(E, \mathbb{R}) \longrightarrow H_1(M, \mathbb{R}) \rightarrow \mathbb{R} \rightarrow 0 \end{aligned}$$

Since f is an bundle morphism, f^* respects the Wang sequence. Hence $f^*|_{H^1} = {}^t C$ and is Anosov. By Poincare duality, $f^*|_{H^2} = C^{-1}$ is Anosov; so does $f^*|_{H_1}$ and $f^*|_{H_2}$. Now $1 - f^*|_{H_1}$ and $1 - f^*|_{H_2}$ are both isomorphism, and the result follows. \square

We have another observation for S^1 -manifolds:

Proposition 4.11 *If M is a S^1 -manifold, then M has the structure of a circle bundle over a S^1 -manifold B , and the Euler class is non-torsion.*

Proof: The bundle structure comes from the circle bundle structure of E . f is a bundle isomorphism of E which covers an Anosov map C of T^2 . The projection of E to T^2 is f -equivariant, so induces a global circle bundle structure on M . The base manifold B is a T^2 -bundle over S^1 with glueing map C , and hence is a S^1 -manifold.

If (M) is torsion, then M will finitely cover $B \times S^1$, and will have $S^1 \times \mathbb{E}$ geometry. This contradicts Theorem 2.1. \square

For N^2 -manifolds, we have analogous results:

Proposition 4.12 *If M is a N^2 -manifold, then M has the structure of a circle bundle over N^2 -manifold, and the Euler class is non-torsion.*

Proof: By Proposition 4.1, M is a T^3 -bundle over S^1 with glueing matrix

$$A = \begin{pmatrix} 1 & \lambda & . \\ 0 & 1 & . \\ 0 & 0 & 1 \end{pmatrix}$$

such that $\lambda \neq 0$, $\neq 0$. By projecting to the last two coordinates, M is expressed as a circle bundle over a N^2 -manifold. If the Euler class is torsion, then M will have $N^2 \times \mathbb{E}$ geometry, which contradicts Theorem 2.1. \square

5 Proof of Diagram 1-1 (without $\mathbb{H}^2 \times \mathbb{H}^2$): Arrows

Let us recall Diagram 1-1:

$$\begin{array}{ccccccc}
 & & \mathbb{H}^2 \times \mathbb{E}^2 & \longrightarrow & \mathbb{H}^2 \times S^2 & & \mathbb{C}\mathbb{P}^2 \\
 & \nearrow & \downarrow & & \downarrow & & \uparrow \\
 \mathbb{H}^3 \times \mathbb{E} & & \mathbb{E}^4 & \longrightarrow & S^2 \times \mathbb{E}^2 & \longrightarrow & S^2 \times S^2 \\
 & \searrow & \nearrow & & \searrow & & \nwarrow \\
 & & S \times \mathbb{E} & & S^3 \times \mathbb{E} & \longleftarrow & N^4 \\
 & \searrow & & & \uparrow & & \\
 \widetilde{SL_2\mathbb{R}} \times \mathbb{E} & \longrightarrow & N \times \mathbb{E} & & S^4_{m,n}, S^4_0, S^4_1 & &
 \end{array}$$

Using the results developed in section 3 and 4, it is now easy to prove the arrows. Our purpose to develop the notion of typical covering manifold is the following statement:

Proposition 5.1 *Let X and Y be geometries. If every typical covering manifold of X dominates some Y -manifold, then $X \rightarrow Y$ holds. \square*

Then we can begin the proof of arrows:

Proposition 5.2 (1) $\mathbb{H}^2 \times \mathbb{E}^2 \rightarrow \mathbb{H}^2 \times S^2 \rightarrow S^2 \times \mathbb{E}^2 \rightarrow S^2 \times S^2 \rightarrow \mathbb{C}\mathbb{P}^2$. (2) $\mathbb{H}^2 \times \mathbb{E}^2 \rightarrow \mathbb{E}^4 \rightarrow S^2 \times \mathbb{E}^2 \rightarrow S^3 \times \mathbb{E}$.

Proof: Use Proposition 5.1 and Table 3-1.

(1) Since high-genus surface dominates low-genus surface, we clearly have $\Sigma_{g_1} \times T^2$ dominates $\Sigma_{g_1} \times S^2$ dominates $T^2 \times S^2$ dominates $S^2 \times S^2$ dominates $\mathbb{C}\mathbb{P}^2$.

(2) It is also clear that $\Sigma_g \times T^2$ dominates $T^2 \times T^2$ dominates $S^2 \times T^2 = S^2 \times S^1 \times S^1$ dominates $S^3 \times S^1$. \square

Proposition 5.3 (1) $\mathbb{H}^3 \times \mathbb{E}$ dominates $\mathbb{H}^2 \times \mathbb{E}^2, S \times \mathbb{E}$ and $\widetilde{SL_2\mathbb{R}} \times \mathbb{E}$. (2) $S \times \mathbb{E} \rightarrow S^2 \times \mathbb{E}^2$. (3) $\widetilde{SL_2\mathbb{R}} \times \mathbb{E} \rightarrow N \times \mathbb{E} \rightarrow S^2 \times \mathbb{E}^2$.

Proof: This is a direct consequence of Corollary 7.5. \square

Proposition 5.4 $N^4, S^4_{m,n}, S^4_0, S^4_1$ dominates $S^3 \times \mathbb{E}$.

Proof: By Proposition 4.1, 4.3, 4.5 and 4.8, manifolds with $N^4, S^4_{m,n}, S^4_0$ or S^4_1 geometry have the structure of a fiber bundle over S^1 . Let M be such a manifold. We can find a section of M and a tubular neighborhood $U \supset \text{section}$ such that the intersection of U with each fiber is homeomorphic to a disk. By contracting the complement of this disk on every fiber, we get a map from M to a S^3 -bundle over S^1 , which must be isomorphic to the trivial bundle $S^3 \times S^1$. \square

To determine which geometry dominates $S^2 \times S^2$, we need to develop a technical lemma:

Lemma 5.1 *There exists $\alpha : M \rightarrow S^2 \times S^2$ with $\deg(\alpha) \neq 0$ iff $\chi_2(M) > 1$ and there is a nonzero element $\beta \in H^2(M, \mathbb{R})$ such that $\beta^2 = 0$.*

Proof: Let α, β be the standard generators of $H^2(S^2 \times S^2, \mathbb{R})$.

If such α exists, then $\alpha^*(\alpha) \alpha^*(\beta) = \alpha^*(\beta) = \deg(\alpha)[M] \neq 0$ and $\alpha^*(\alpha)^2 = \alpha^*(\beta)^2 = 0$. This implies that $\alpha^*(\alpha)$ and $\alpha^*(\beta)$ are linearly independent, and thus $\chi_2(M) > 1$. α is our desired element.

Conversely: $\beta^2 = 0$ means that the intersection form of M is not definite. Hence there exists another nonzero element $\gamma \in H^2(M, \mathbb{R})$ such that $\gamma^2 = 0$ and $\beta \neq \gamma$. β induces a map $\alpha_x : M \rightarrow \mathbb{C}P^\infty$ which by cellular approximation, can be homotoped to a map $\alpha'_x : M \rightarrow \mathbb{C}P^2$. Since $\beta^2 = 0$, we have $\deg(\alpha'_x) = 0$. This implies that α'_x can be deformed to a map $\alpha''_x : M \rightarrow S^2$. Similarly, we can define α''_y . Consider $\alpha = \alpha''_x \times \alpha''_y : M \rightarrow S^2 \times S^2$. We have $\deg(\alpha)[M] = \beta \gamma$ hence $\deg(\alpha) \neq 0$. \square

Proposition 5.5 $N^4 \rightarrow S^2 \times S^2$.

Proof: By Proposition 4.2, a N^4 manifold has $\chi_2 = 2$. \square

Now the arrows in Diagram 1-1 are all proven.

6 Proof of Diagram 1-1: Non-arrows (Part 1)

In this section, we prove a part of the non-arrows. Those we wish to prove cannot be derived from lower-dimensional results, and must be proved independently. The proof depends on several technical lemmas:

Lemma 6.1 *If $\alpha : M \rightarrow N$ is of nonzero degree, then the image of $\alpha_* : \pi_1(M) \rightarrow \pi_1(N)$ is of finite index in $\pi_1(N)$.*

Proof: Let $d = \deg(\alpha)$ and $H = \text{Im}(\alpha_*)$. If $[\pi_1(N) : H] > d$, then there exists a subgroup $H' \subset \pi_1(N)$ such that $\infty > [\pi_1(N) : H'] > d$. Let N' be the (finite) covering space of N such that $\pi_1(N') = H'$, and denote by β the projection map. By knowledge of covering spaces, there exists a map $\alpha' : M \rightarrow N'$ such that $\alpha = \beta \circ \alpha'$. Calculating the degree we get a contradiction. \square

Corollary 6.1 *If $\chi_1 < \chi_2$, then Σ_{g_1} does not dominate Σ_{g_2} . If S is simply-connected, then $S \times \Sigma_{g_1}$ does not dominate $S \times \Sigma_{g_2}$.*

Proof: If $\alpha : \Sigma_{g_1} \rightarrow \Sigma_{g_2}$ has nonzero degree, then by Lemma 6.1, there exists a finite cover $\Sigma_{g_3} \rightarrow \Sigma_{g_2}$ and a lifted map $\tilde{\alpha} : \Sigma_{g_1} \rightarrow \Sigma_{g_3}$, such that $\tilde{\alpha}_* : \pi_1(\Sigma_{g_1}) \rightarrow \pi_1(\Sigma_{g_3})$ is surjective. Obviously $\chi_3 > \chi_2$; and $\tilde{\alpha}_* : H_1(\Sigma_{g_1}, \mathbb{Z}) \rightarrow H_1(\Sigma_{g_3}, \mathbb{Z})$ is also surjective. This is impossible.

The second conclusion holds for the same reason. \square

A $K(G, 1)$ is a CW complex which has contractible universal cover and has fundamental group G ; the homotopy type of $K(G, 1)$ is unique. A detailed introduction of $K(G, 1)$ can be found in section 1.B of [3]. If X is a geometry which is not $S^4, S^3 \times \mathbb{E}^2, S^2 \times S^2, S^2 \times \mathbb{E}^2$ or $S^2 \times \mathbb{H}^2$, then every X -manifold is a $K(G, 1)$.

Lemma 6.2 ([3, Proposition 1B.9]) *Let M be a connected CW complex and let N be a $K(G, 1)$. Then every homomorphism $\pi_1(M) \rightarrow \pi_1(N)$ is induced by a map $M \rightarrow N$ which is unique up to homotopy. \square*

The fundamental criterion for non-arrows is the following statement. This criterion is much stronger than the definition of non-domination, but still grabs the essence.

Proposition 6.2 *Assume that X, Y are two geometries, and \mathcal{A} (resp. \mathcal{B}) is a set of typical covering manifold for X (resp. Y). If for every $M_1 \in \mathcal{A}$ and $M_2 \in \mathcal{B}$, M_1 does not dominate M_2 , then $X \not\rightarrow Y$.*

Proof: Pick any X -manifold M_1 , and assume that M_1 dominates some Y -manifold M_2 . M_2 is finitely covered by some $M'_2 \in \mathcal{B}$. The pull back $M'_1 = M_1 \times_{M_2} M'_2$ is a finite cover of M_1 , and is covered by some $M''_1 \in \mathcal{A}$. The composition map $M''_1 \rightarrow M_1 \rightarrow M_2$ is of nonzero degree, which contradicts the assumption in our proposition. \square

Now we start proving non-arrows. We do not need to prove every non-arrow, since the domination relation is transitive. For example, if we wish to prove that N^4 does not dominate any other geometries except those shown in the diagram, we only need to show that it does not dominate $S^2 \times \mathbb{E}^2, S^4_0, S^4_{m,n}$ and S^4_1 . For another example, if we have shown that $\mathbb{H}^2 \times \mathbb{E}^2, \mathbb{H}^3 \times \mathbb{E}, S^4_0, S^4_{m,n}$ and S^4_1 does not dominate N^4 , then there would be no extra arrows pointing toward N^4 .

With a little effort, we can check that the propositions proved below, combined with Corollary 8.1, are sufficient to imply all non-arrows.

Proposition 6.3 (1) $\mathbb{C}\mathbb{P}^2 \not\rightarrow S^2 \times S^2, \mathbb{C}\mathbb{P}^2 \not\rightarrow S^3 \times \mathbb{E}$. (2) $S^2 \times S^2 \not\rightarrow S^3 \times \mathbb{E}, S^3 \times \mathbb{E} \not\rightarrow \mathbb{C}\mathbb{P}^2$.

Proof:

(1) Note that $\mathbb{C}\mathbb{P}^2$ is the only $\mathbb{C}\mathbb{P}^2$ -manifold — a consequence of the Lefschetz fixed-point formula. The first conclusion is from Lemma 5.1 and $\chi_2(\mathbb{C}\mathbb{P}^2) = 1$. The second conclusion is from $H^1(\mathbb{C}\mathbb{P}^2) = 0$ so that $\mathbb{C}\mathbb{P}^2$ does not dominate $S^3 \times S^1$. By Table 3-1(3), $S^3 \times S^1$ is the typical covering manifold of $S^3 \times \mathbb{E}$.

(2) Clearly $S^2 \times S^2$ does not dominate $S^3 \times S^1$ and $S^3 \times S^1$ does not dominate $\mathbb{C}\mathbb{P}^2$. \square

Corollary 6.4 $S^2 \times S^2$ and $S^3 \times \mathbb{E}$ dominates no other geometry except those indicated in the domination diagram. \square

Proposition 6.5 (1) $S^4_{m,n}, S^4_0, S^4_1$ does not dominate $\mathbb{C}\mathbb{P}^2$. (2) N^4 does not dominate $S^2 \times \mathbb{E}^2$. (3) $N^4, S^4_{m,n}, S^4_0, S^4_1$ does not dominate each other.

Proof:

(1) By Proposition 4.4, 4.7, 4.10, $S^4_{m,n}, S^4_0, S^4_1$ -manifolds have $\chi_2 = 0$.

(2) Let M be any N^4 -manifold. Again we look back at the remark below Proposition 4.2 and use the notations defined there. $H_3(M, \mathbb{R})$ is generated by ϵ , which is represented by the T^3 -fiber, and ϕ , which is a subbundle. The intersection of these two chains is a hyperplane in the T^3 fiber, which is spanned by $\{x, y\}$. Denote this chain by γ . From the proof of proposition 4.2, we can know that γ lies in the image of $1 - A$ (here the coefficient is real numbers), and thus projects to zero in $H_2(M, \mathbb{R})$. Now denote by $\hat{\epsilon}, \hat{\phi} \in H^1(M, \mathbb{R})$

the Poincaré dual of ϵ, ϕ , then we have $\hat{\epsilon} = 0 \in H^2(M, \mathbb{R})$. Since they generate H^1 , for any two elements $\mu, \nu \in H^1(M, \mathbb{R})$ we have $\mu \nu = 0$.

Suppose that $f : M \rightarrow S^2 \times T^2$ is of nonzero degree. Assume that $\alpha, \beta \in H^1(S^2 \times T^2, \mathbb{R})$ and $\gamma \in H^2(S^2 \times T^2, \mathbb{R})$ are standard generators, then we have $f^* \gamma = \pm[S^2 \times T^2]$ and $f^*(\alpha \beta) = (f^* \alpha)(f^* \beta) = 0$, thus $\deg(f) = 0$. Finally use Proposition 5.6 and Table 3-1(4).

(3) By Proposition 4.1, 4.3, 4.7 and 4.10, if M is a manifold of $N^4, S^4_{m,n}, S^4_0$ or S^4_1 type, then $\pi_1(M)$ has the form $\pi_1(M) = N \rtimes_A \mathbb{Z}$, where N is \mathbb{Z}^3 or the fundamental group of a N -manifold, and $A \in \text{Aut}(N)$.

Let M_1 be a S^4_1 -manifold, M_2 be of $N^4, S^4_{m,n}$ or S^4_0 type, and suppose $f : M_1 \rightarrow M_2$ be of nonzero degree. By Lemma 6.1, after taking a finite cover of M_2 we can assume that $f_* : \pi_1(M_1) \rightarrow \pi_1(M_2)$

Proof: Let M be a typical covering manifold of any of these geometries. M has the form $N \times S^1$. $\pi_1(M)$ has non-trivial center, which must be mapped to zero by $*$. By Lemma 6.2, $*$ can be homotoped to compress the S^1 factor of M , hence has zero degree. \square

Proposition 6.9 $\mathbb{H}^3 \times \mathbb{E}$ does not dominate $N^4, S^4_{m,n}, S^4_0$ or S^4_1 .

Proof: Assume that $M_1 = N \times S^1$ is a typical covering manifold of $\mathbb{H}^3 \times \mathbb{E}$, and M_2 is any $N^4, S^4_{m,n}, S^4_0$ or S^4_1 -manifold. $f: M_1 \rightarrow M_2$ is a map of nonzero degree. By taking a finite cover of M_2 , we assume that f_* is surjective on fundamental groups.

For $S^4_{m,n}$ and S^4_0 case: By Proposition 4.3 and 4.5, $\pi_1(M_2)$ has trivial center. But $\pi_1(M_1)$ has nontrivial center, thus we have a contradiction.

For N^4 and S^4_1 case: By Proposition 4.12 and 4.11, M_2 is a circle bundle: $S^1 \rightarrow M_2 \rightarrow F$ where F is of N^3 or S^3 type. Thus $\pi_1(M_2)$ fits into an exact sequence: $0 \rightarrow \mathbb{Z} \rightarrow \pi_1(M_2) \rightarrow \pi_1(F) \rightarrow 0$. Furthermore, \mathbb{Z} is the center of $\pi_1(M_2)$; and f_* must send $\pi_1(S^1) \in \pi_1(M_1)$ isomorphically onto \mathbb{Z} . Regarding M_1 as a trivial circle bundle over F , and by Lemma 6.2, f is homotopic to a bundle morphism f' . f' induces a map $f'_*: N \rightarrow F$. f'_* is of nonzero degree since f_* is (otherwise, f can be homotoped to a map from N to the 2-skeleton of F). By the homotopy lifting property, f' can be homotoped to the 3-skeleton of M_2 .

Note that f' induces a bundle isomorphism between M_1 and f'^*M_2 , so we have $f'^*(M_2) = (f'^*M_2) = (M_1) = 0$ where e denotes Euler class. By Proposition 4.12 and 4.11, (M_2) is non-torsion, hence is nonzero in $H^2(F, \mathbb{R})$. By Poincare duality, there exists $\lambda \in H^1(F, \mathbb{R})$ such that $(M_2) = \lambda[F]$ ($\lambda \neq 0$). However, $\lambda^*([F]) = \lambda^*((M_2)) = \lambda^*(\lambda)^*((M_2)) = 0$, so $\deg(\lambda) = 0$. \square

7 Product Geometry Splitting

Inspired by Table 3-1 and its proof, we are able to prove the following generalized conclusion:

Proposition 7.1 Let X be any geometry of dimension 2, 3 or 4, and k be any positive integer. Then every $X \times \mathbb{E}^n$ -manifold is finitely covered by $N \times T^n$ where N is a X -manifold.

Proof: Discuss case by case. First we fix our notation: G_0 is the structure group of $X \times \mathbb{E}^n$, G is the structure group of X . Γ is a discrete subgroup of G_0 acting freely on X such that $M = \Gamma \backslash X$ is compact. We have explained in section 3 that Γ is a lattice in G_0 .

(1) $X = S^k$ ($k = 2, 4$):

$G_0 = SO(k+1) \times I^0(\mathbb{E}^n)$. Odd-dimensional orthogonal operator has fixed-point, hence the projection map $\Gamma \rightarrow I^0(\mathbb{E}^n)$ is an isomorphism. M is expressed as a S^k -bundle over a Euclidean k -manifold. Passing to a finite cover we can assume that the base manifold B is T^n , and now $\Gamma \cong \mathbb{Z}^n$. Let $\gamma_i = (\phi_i, \nu_i)$ be a basis of Γ where $\phi_i \in SO(k+1)$.

As all the ϕ_i commutes, they are simultaneously diagonalized to the form

$$\begin{pmatrix} \cos \theta_i & -\sin \theta_i & & & \\ \sin \theta_i & \cos \theta_i & & & \\ & & & & \\ & & & & \\ & & & & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \cos \theta_i & -\sin \theta_i & & & \\ \sin \theta_i & \cos \theta_i & & & \\ & & \cos \theta_i & -\sin \theta_i & \\ & & \sin \theta_i & \cos \theta_i & \\ & & & & 1 \end{pmatrix}$$

This means that ϕ_i has common fixed point on S^k . This fixed point induces a section of M over B , and thus M is a trivial circle bundle over B . \square

The case $X = S^2 \times S^2$ can be proved in the same way.

(2) $X = S^3$:

M is a smooth orbifold fibration over Euclidean n -orbifold with general fiber a S^3 -manifold. Taking covering spaces first with respect to the base orbifold and then with respect to the fiber (the reader can check this), we can show that M is finitely covered by a S^3 -bundle over T^n .

Next, the situation is analogous to (1). The difference is that ϕ_i does not necessarily has fixed point. But we can still diagonalize ϕ_i to the form:

$$\begin{pmatrix} \cos \theta_i & -\sin \theta_i & & & \\ \sin \theta_i & \cos \theta_i & & & \\ & & \cos \theta_i & -\sin \theta_i & \\ & & \sin \theta_i & \cos \theta_i & \\ & & & & 1 \end{pmatrix}$$

Let us fixed the basis of \mathbb{R}^4 such that ϕ_i has the form above. The following map induces a section of M over T^n :

$$\sum \lambda_i \mathbf{e}_i \mapsto \frac{1}{\sqrt{2}}(\cos(\sum \lambda_i \theta_i), \sin(\sum \lambda_i \theta_i), \cos(\sum \lambda_i \theta_i), \sin(\sum \lambda_i \theta_i))$$

Hence M is a trivial S^3 -bundle over T^n . \square

The same method applies to the case $X = \mathbb{C}\mathbb{P}^2$, where $G_0 = SU(3) \times I^{-1}(\mathbb{E}^n)$.

(3) $X = \mathbb{H}^2, \mathbb{H}^3, \mathbb{H}^2 \times \mathbb{H}^2$:

In this case, $G_0 = G \times I^{-1}(\mathbb{E}^n)$. G is semisimple. By Lemma 3.4 (for $n = 1, 2$) and Lemma 7.1 below (for $n > 2$; the group S is $G \times SO(n)$ here), $\Gamma' = \Gamma \cap I^{-1}(\mathbb{E}^n)$ is a lattice (because $\Gamma \cap \mathbb{R}^n$ is a lattice), and the image H of the projection map $\Gamma \rightarrow G$ is also a lattice.

If Γ' contains only translation, then M is a Seifert T^n -bundle over H/X . Passing to a finite cover we can assume that H/X is smooth. The monodromy lies in $SO(n) \cap SL(\Gamma')$. This is a finite group (since $SL(\Gamma')$ is discrete, and any $\gamma \in SO(n)$ has $\|\gamma\|_2 = 1$). The rest of proof is the same as Table 3-1(5).

Suppose that Γ' contains a non-translation element γ . The translation subgroup of Γ' (denote by Γ'') is of finite index. Denote by K the image of projection: $\Gamma \rightarrow I^{-1}(\mathbb{E}^n)$. K fit into an exact sequence: $0 \rightarrow K' \rightarrow K \rightarrow K'' \rightarrow 0$ where $K' = K \cap \mathbb{R}^n$ and $K'' = K/K' \subset SO(n)$. From conjugation actions (K' on Γ' , and K'' on Γ''), we can show that Γ'' is of finite index in K' , and $K'' \subset SO(n) \cap SL(\Gamma'')$. Hence K is discrete. Then it is easy to show that $\Gamma \cap G$ is a lattice in G , and K/Γ' is finite. Finally, $\Gamma' \times (\Gamma \cap G)$ is a finite-index subgroup in Γ ; M is finitely covered by $(\mathbb{E}^n\text{-manifold}) \times (X\text{-manifold})$. \square

Lemma 7.1 ([9, Corollary 8.28]) *Let G be a connected Lie group, R be its radical. Let S be a semisimple subgroup of G such that $G = SR$. Denote by σ the conjugation action of S on R . Assume that $\ker(\sigma)$ has no compact factor in its identity component. Then if Γ is a lattice in G , $\Gamma \cap R$ is a lattice in R . \square*

(4) $X = \widetilde{SL_2\mathbb{R}}$:

$G_0 = (\widetilde{SL_2\mathbb{R}} \times_{\mathbb{Z}} \mathbb{R}) \times I^{-1}(\mathbb{E}^n)$. Denote $N = \mathbb{R} \times I^{-1}(\mathbb{E}^n)$, in which the \mathbb{R} factor is the center of $I^{-1}(\widetilde{SL_2\mathbb{R}})$. We claim that $\Gamma' = \Gamma \cap N$ is a lattice. The remaining part of the proof works in the same way as in (3) and Table 3-1(7).

The case $n = 1, 2$ follows from Lemma 3.4; it suffices to prove the case $n > 2$. The proof here is based on the proof of ([9, Theorem 8.27]). Denote $R = \mathbb{R} \times \mathbb{R}^n$, and by π_1 the projection map $G_0 \rightarrow G_0/R = I^{-1}(\mathbb{H}^2) \times SO(n)$. Denote by π_2 the projection map $I^{-1}(\mathbb{H}^2) \times SO(n) \rightarrow I^{-1}(\mathbb{H}^2)$. Let U be the closure of $\pi_1(\Gamma)$. According to ([9, Theorem 8.24]), the identity component U^0 is solvable. The group $V = \pi_2(\pi_1(\Gamma)) \subset I^{-1}(\mathbb{H}^2)$ has property (S) as defined in Chapter 5, [9]. For the same reason as in ([9, Theorem 8.27]), $\pi_2(U^0)$ is a normal subgroup of $I^{-1}(\mathbb{H}^2)$, and is trivial. \bar{V}^0 has three possibilities: ① is trivial, then our claim is proven. ② \bar{V}^0 is a proper closed subgroup of $I^{-1}(\mathbb{H}^2)$. Its Lie algebra \mathfrak{v} is preserved by the adjoint action of V . By [9, Corollary 5.16(i)], \mathfrak{v} is normal in $\mathfrak{iso}(\mathbb{H}^2)$, and is trivial. ③ $\bar{V}^0 = I^{-1}(\mathbb{H}^2)$, i.e. V is dense. In addition that $SO(n)$ is compact, given any $v \in I^{-1}(\mathbb{H}^2)$, there exists a sequence $v_i \in \pi_1(\Gamma)$ converging to (v, \cdot) for some $\cdot \in SO(n)$. This means that $\pi_2(U) = I^{-1}(\mathbb{H}^2)$. On the other hand, since $U^0 \subset SO(n)$, $\pi_2(U) = \pi_2(\pi_1(\Gamma))$. Γ is finitely-generated, this is impossible. \square

(5) $X = \mathbb{H}^2 \times S^2$:

M is a S^2 -bundle over a $\mathbb{H}^2 \times \mathbb{E}^n$ -manifold, hence is covered by a S^2 -bundle over $\Sigma_g \times T^n$. Let $e_i, e_i (1 \leq i \leq g)$, $e_i (1 \leq i \leq n)$ be the standard generators of $\Sigma_g \times T^n$. Since $[e_i, e_j] = [e_i, e_j] = [e_i, e_j] = 1$, for the same reason as in (1), M has a section and is a trivial bundle. \square

(6) $X = N$:

The proof is quite similar to Table 3-1(8). In this case, $G_0 = (N \rtimes S^1) \times (\mathbb{R}^n \rtimes SO(n))$. From Lemma 7.1 (for $n > 2$) and Lemma 3.2 (for $n = 1, 2$) we know that $\Gamma \cap (N \times \mathbb{R}^n)$ is a lattice. The quotient group is finite; taking a finite cover we can assume that $\Gamma \subset N \times \mathbb{R}^n$. Working as in Table 3-1(8), we can then prove that $\Gamma \cap (\mathbb{R} \times \mathbb{R}^n)$ is a lattice. Hence M is a T^{n+1} -bundle over T^2 . The monodromy is again trivial, and finally M splits as a product of T^n and a N -manifold. \square

The remaining cases are $S^3, N^4, S^4_0, S^4_{m,n}, S^4_1$. They are their own structure group, and are solvable. In these cases, $G_0 = G \times I^{-1}(\mathbb{E}^n) = (G \times \mathbb{R}^n) \times SO(n)$. Lemma 7.1 applies to all cases for $n > 2$: Denote $G' = G \times \mathbb{R}^n$, then $\Gamma \cap G'$ is a lattice. The quotient is a lattice in $SO(n)$, hence is finite. Replacing M by a finite cover, in the following paragraphs we assume that $\Gamma \subset G'$.

(When $n = 2$, use Lemma 3.2 instead and we can prove the same conclusion after some adjustment of the proofs below.)

(7) $X = S^3$:

$\Gamma \subset S^3 \times \mathbb{R}^n$ is a lattice. Denote $N = V_1 \times V_2$ where $V_1 = \mathbb{R}^2$ is the nilradical of S^3 , and $V_2 = \mathbb{R}^n$. By Lemma 3.2, $\Gamma \cap N$ is a lattice. Analogous to Table 3-1(9), we can show

that there exists a finite-index sublattice $\Gamma' = \Gamma_1 \times \Gamma_2 \subset \Gamma$ such that Γ_i are lattices in V_i . Thus M is finitely covered by $(S^4\text{-manifold}) \times T^n$. \square

(8) $X = S^4_0, S^4_{m,n}$:

We have Proposition 4.3, 4.5. Then the proof is analogous to (7). \square

(9) $X = N^4$:

Now $\Gamma \subset N^4 \times \mathbb{R}^n$. Denote by $(x_1, x_2, x_3; \dots)$ the standard coordinates in N^4 . The commutator subgroup is $N = \mathbb{R}(x_1, x_2)$. By Lemma 3.3, $\Gamma \cap N$ is a lattice. According to the conjugation relation (see section 4, part 1), we can assume that $(x_1, 0, 0; 0), (x_1, x_2, 0; 0)$ is a basis of $\Gamma \cap N$. $\Gamma' = \Gamma/(\Gamma \cap N) \subset \mathbb{R}^{2+n}$ is also a lattice. Assume that $\gamma_i = (s_{i1}, s_{i2}, s_{i3}; \dots; \theta_{i1} \dots \theta_{in}) \in \Gamma$ are $n+2$ elements that projects to a basis of Γ' . Without losing generality, we can assume that $s_{13} s_{21} - s_{23} s_{11} \neq 0$. By some calculation,

$$[\gamma_i, \gamma_j] = (*, s_{j3} s_{i1} - s_{i3} s_{j1}, 0; 0; 0 \dots 0)$$

Thus $s_{ij} = s_{j3} s_{i1} - s_{i3} s_{j1}$ is an integral multiple of s_{21} . For $i = 3, 4, \dots, n+2$, the element $\gamma'_i = \frac{s_{12}/h_2}{s_{i1}} \frac{-s_{2i}/h_2}{s_{i1}} \frac{s_{1i}/h_2}{s_{i1}}$ has the form $(s_{i1}, s_{i2}, 0; 0; * \dots *)$. Assume $s_{i1} \neq 0$; calculating $\gamma'_i \gamma_{i-1}^{-1}$ and $\gamma'_i \gamma_{i-1}^{-1}$, we find that s_{i2} and s_{i1} are all integral multiples of s_{21} . Thus the ratio between s_{i2} and s_{i1} are all rational. Define $\gamma''_i = \frac{\gamma'_i}{s_{i1}} + \dots$ ($\dots \in \mathbb{Z}$), they will have the form $\gamma''_i = (*, 0, 0; 0; * \dots *)$.

Denote by Γ_1 the lattice generated by $\gamma_1, \gamma_2, \dots, \gamma_{n+2}$. It is a finite-index sublattice of Γ . $\gamma_3 \dots \gamma_{n+2}$ freely generates an abelian subgroup $U_1 \subset \Gamma_1$. γ_1, γ_2 generates a subgroup U_2 which is isomorphic to a lattice in N^4 . Clearly $\Gamma_1 = U_1 \times U_2$. Hence M is a product of N^4 -manifold and T^n . \square

(10) $X = S^4_1$:

$\Gamma \subset S^4_1 \times \mathbb{R}^n$ is a lattice. In section 4 (part 4), we introduced the structure of S^4_1 . By Lemma 3.2, $\Gamma' = \Gamma \cap (N^4 \times \mathbb{R}^n)$ is a lattice. From our results in (6) and Table 3-1(8), we can assume that Γ' is generated by $\gamma_1, \gamma_2, \dots, \gamma_n$, where $\gamma_i \in (N^4), [\gamma_i, \gamma_j] = \gamma_i^k, \theta_i \in (N^4) \times \mathbb{R}^n$. $\Gamma/\Gamma' \subset S^4_1/N^4 = \mathbb{Z}$ is also a lattice, and we suppose that $\gamma_{n+1} \in \Gamma$ projects to the generator of it. Now $\theta_1 \dots \theta_n$ freely generates an abelian group V_1 . $\gamma_1, \gamma_2, \dots, \gamma_n$ generates a group V_2 which is isomorphic to a lattice in S^4_0 . Also, $\Gamma = V_1 \times V_2$. The product structure follows. \square

Finally, $X = \mathbb{E}^k$ follows from Bieberbach Theorem. \square

Corollary 7.2 *Suppose the dimension of X , n are both 3 or 4. Then $X \rightarrow \mathbb{E}^n$ implies $X \times \mathbb{E}^n \rightarrow \mathbb{E}^n \times \mathbb{E}^n$. \square*

It is natural to may ask whether the converse holds. The answer is affirmative.

Proposition 7.3 *Suppose M_1, M_2 are two manifolds of the same dimension. M_2 is a $K(G, 1)$ and $\pi_1(M_2)$ is torsion-free. If $M_1 \times S^1$ dominates $M_2 \times S^1$, then some finite cover of M_1 dominates M_2 .*

Proof: Let $f: M_1 \times S^1 \rightarrow M_2 \times S^1$ be a nonzero degree map. Denote by γ_i the generator of $\mathbb{Z} = \pi_1(S^1) \subset \pi_1(M_i \times S^1)$. $f_*(\gamma_1)$ is nonzero: otherwise from Lemma 6.2, f can be

homotoped to factor through $M_1 \times \{ \}$, and cannot have nonzero degree. Since $\pi_1(M_2)$ is torsion-free, we have $\ker(\ast) \subset \pi_1(M_1)$. Let $H \subset \pi_1(M_2 \times S^1)$ be the image of \ast ; clearly H has finite index. Assume that the projection map $H \rightarrow \mathbb{Z}_2$ is surjective (otherwise we replace the S^1 factor in $M_2 \times S^1$ by a finite cover of it).

Let n be the smallest integer such that $\frac{1}{2} = \frac{n}{2}$ lies in H . Denote $I = H \cap \pi_1(M_2)$. Then $I \times \mathbb{Z}'_2$ has finite index in H (Lemma 3.5). On the other hand: define $L = \pi_1(M_1)/\ker(\ast)$, then $H = L \times \mathbb{Z}(\ast(1))$.

If $\ast(1) \neq \frac{1}{2}$, then we use the following method to reduce to the case $\ast(1) = \frac{1}{2}$. First we claim that $H' = (I \cap L) \times (\mathbb{Z}(\ast(1)) \times \mathbb{Z}'_2)$ is of finite index in H . This is clearly a direct product. We have an exact sequence $1 \rightarrow IL/(I \cap L) \rightarrow H/(I \cap L) \rightarrow H/IL \rightarrow 1$. IL contains L and is not equal to L , thus is of finite index in H . There exists $a, b \in \mathbb{Z}$ such that $\ast(1) \in IL$, $\frac{1}{2} \in IL$. $IL/(I \cap L)$ is isomorphic to $(IL/I) \times (IL/L)$, and $\ast(1), \frac{1}{2}$ generates a finite-index subgroup of it. Our claim then follows.

Then we take a finite cover of $M_1 \times S^1$ with respect to $\ast^{-1}(H')$. Since $\ast(1) \in H'$, the covering map has the form $\ast \times \ast : M'_1 \times S^1 \rightarrow M_1 \times S^1$. $K(H')$,

①: $S^2 \times S^2 \times \mathbb{E}^n$ and $S^3 \times \mathbb{E}^{n+1}$ does not dominate each other. It suffices to prove
 $S^2 \times S^2 \times T^n$ and $S^3 \times T^{n+1}$ does not dominate each other. The notation

(5) $\mathbb{H}^2 \times \mathbb{E}, \widetilde{SL_2\mathbb{R}}$ does not dominate $S^1 \times \mathbb{E}$. This is because the fundamental group of $S^1 \times \mathbb{E}$ -manifold has trivial center. \square

Corollary 8.1 *Let X_i be any 3-dimensional geometries. Except for those indicated in Diagram 1-1, there is no more arrows between $X_i \times \mathbb{E}$ and $X_j \times \mathbb{E}$. \square*

With Corollary 8.1 and the propositions in section 6, we have completed the proof of non-arrows in Diagram 1-1.

9 Results on $\mathbb{H}^2 \times \mathbb{H}^2$

In this section, we treat the last case: $\mathbb{H}^2 \times \mathbb{H}^2$. Our previous results are mainly based on virtual splitting. However, not all $\mathbb{H}^2 \times \mathbb{H}^2$ -manifold virtually split into a product of surfaces.

An $\mathbb{H}^2 \times \mathbb{H}^2$ -manifold M is called **reducible** if it is finitely covered by $\Sigma_{g_1} \times \Sigma_{g_2}$. Otherwise it is called **irreducible**. The fundamental group of a reducible manifold is commensurable with a product of Fuchsian groups. The fundamental group of an irreducible manifold is arithmetic, in the sense of [4, §9.5].

We separate $\mathbb{H}^2 \times \mathbb{H}^2$ -manifolds into two classes: reducible ones and irreducible ones. Manifolds belonging to different classes are not the covering of each other. The dominating ability of these two classes of manifolds are quite different.

Reducible $\mathbb{H}^2 \times \mathbb{H}^2$ clearly dominates $\mathbb{H}^2 \times \mathbb{E}^2$. For non-domination, we have the following result:

Proposition 9.1 *$\Sigma_{g_1} \times \Sigma_{g_2}$ does not dominate any $S^1 \times \mathbb{E}, N^4, S^4_{m,n}, S^4_0, S^4_1$ -manifold, as well as any typical covering $N^4 \times \mathbb{E}$ -manifold.*

Proof:

Suppose that $\gamma: M_1 \rightarrow M_2$ has nonzero degree, where $M_1 = \Sigma_{g_1} \times \Sigma_{g_2}$, M_2 is any $S^1 \times \mathbb{E}, N^4, S^4_{m,n}, S^4_0$ or S^4_1 -manifold, or any typical covering manifold of $N^4 \times \mathbb{E}$. By Lemma 6.1, we can take a finite cover of M_2 such that γ_* is surjective on fundamental groups. Proposition 3.1 guarantees that this causes no problem in the case of $N^4 \times \mathbb{E}$. By Proposition 3.1, 3.2, 4.1, 4.3, 4.5 and 4.8, $\pi_1(M_2)$ has the form $1 \rightarrow N \rightarrow \pi_1(M_2) \rightarrow \mathbb{Z} \rightarrow 0$ where N is abelian or nilpotent.

Writing $\pi_1(M_2)$ in the form $\pi_1(M_2) = N \rtimes_A \mathbb{Z}$, the action A is described in the conclusions or proofs of the Propositions. We do not use the concrete form of A here, only note that A is non-trivial.

Consider the subgroup $\gamma_*^{-1}(N) \subset \pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}) = \pi_1(M_1)$, and denote by H_i the image of the projection map $\gamma_*^{-1}(N) \subset \pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}) \rightarrow \pi_1(\Sigma_{g_i})$. There are two cases:

(1): One of the H_i , say H_1 , is of infinite index in $\pi_1(\Sigma_{g_1})$. But there is an exact sequence $1 \rightarrow \gamma_*^{-1}(N) \rightarrow \pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}) \rightarrow \mathbb{Z} \rightarrow 0$ induced from the exact sequence of $\pi_1(M_2)$. Now $\gamma_*^{-1}(N) \subset H_1 \times \pi_1(\Sigma_{g_2})$, and the only possibility is that $\gamma_*^{-1}(N) = H_1 \times \pi_1(\Sigma_{g_2})$ and $\pi_1(\Sigma_{g_2})/H_1 = \mathbb{Z}$. Therefore, $\pi_1(\Sigma_{g_2})$ is mapped by γ_* into N .

In the cases of $N^4 \times \mathbb{E}, S^4 \times \mathbb{E}, N^4, S^4_{m,n}$ and S^4_0 , $N = \mathbb{Z}^3$. By Lemma 6.2, γ is homotopic to a composed map $\gamma': \Sigma_{g_1} \times \Sigma_{g_2} \rightarrow \Sigma_{g_1} \times T^3 \rightarrow M_2$; we denote the latter map $\Sigma_{g_1} \times T^3 \rightarrow M_2$ by γ'' . In the case of $N^4 \times \mathbb{E}$, the center of $\pi_1(M_2)$ is isomorphic to \mathbb{Z}^2 .

Hence by a further homotopy, γ compresses one S^1 factor of T^3 and thus factors through $\Sigma_{g_1} \times T^2$. Proposition 7.3 and 2.3 now tells us that $\deg(\gamma)$ must be zero. In other cases, the center of $\pi_1(M_2)$ is either trivial or is isomorphic to \mathbb{Z} , γ_* must send $\pi_1(T^3) = \mathbb{Z}^3$ to zero or \mathbb{Z} . Anyway, γ is homotopic to a map $\Sigma_{g_1} \times T^3 \rightarrow \Sigma_{g_1} \times S^1 \rightarrow M_2$, which has zero degree for dimension reason. In all situations, we have $\deg(\gamma) = 0$.

In the case of S^4 , N is the fundamental group of a N -manifold E and can be presented as $N = \langle \gamma, \gamma' : [\gamma, \gamma'] = 1, [\gamma, \gamma'] = k \rangle$. γ generates the center of N , which is also the center of $\pi_1(M_2)$. γ_* factors through N , hence by Lemma 6.2 γ is homotopic to a composed map $\gamma' : \Sigma_{g_1} \times \Sigma_{g_2} \rightarrow \Sigma_{g_1} \times E \rightarrow M_2$; we denote the latter map by γ' . To avoid confusion, we let $\pi_1(E)$ be presented as $\pi_1(E) = \langle \gamma', \gamma'' : [\gamma', \gamma''] = 1, [\gamma', \gamma''] = k \rangle$, which is isomorphic to N . By construction, γ_* maps γ' to γ . Obviously $\pi_1(\Sigma_{g_1} \times E)/\mathbb{Z} \cong \pi_1(\Sigma_{g_1}) \times \mathbb{Z}^2$. By Proposition 4.11, $\pi_1(M_2)/\mathbb{Z} := L$ is isomorphic to the fundamental group of a S^4 -manifold. Taking this quotient on both sides, γ_* induces a homomorphism $\gamma'_* : \pi_1(\Sigma_{g_1}) \times \mathbb{Z}^2 \rightarrow L$. However, being the fundamental group of a S^4 -manifold, L has trivial center. Hence $\mathbb{Z}^2 \subset \ker(\gamma'_*)$. Returning to γ_* , this means that γ_* maps γ' and γ'' into \mathbb{Z} . Then $k = \gamma_*([\gamma', \gamma'']) = \gamma_*([\gamma', \gamma'']) = [\gamma_*([\gamma', \gamma'']), \gamma_*([\gamma', \gamma''])] = 1$ which is a contradiction.

(2): Both H_i has finite index in $\pi_1(\Sigma_{g_i})$. Take a finite cover $M'_1 \rightarrow M_1$ corresponding to $H_1 \times H_2$, and still denote by γ the composed map $M'_1 \rightarrow M_2$. $\gamma_*(\pi_1(M'_1))$ still contains $N \in \pi_2(M_2)$. We take a further cover $M'_2 \rightarrow M_2$ corresponding to the image $\gamma_*(\pi_1(M'_1))$. Now $\pi_1(M'_2)$ fits into the exact sequence $1 \rightarrow N \rightarrow \pi_1(M'_2) \rightarrow \mathbb{Z} \rightarrow 0$, and we have an induced map $\gamma' : M'_1 \rightarrow M'_2$ which has the same degree as γ . γ'_* is still surjective on fundamental groups. M'_1 still has the form of $\Sigma_{g'_1} \times \Sigma_{g'_2}$. $\gamma'^{-1}(N)$ is not changed. We now replace M_1 by M'_1 and M_2 by M'_2 , and inherit all the notations defined above.

The difference is that now we have $H_i = \pi_1(\Sigma_{g_i})$. Define $I_i = \gamma'^{-1}(N) \cap \pi_1(\Sigma_{g_i})$. Clearly I_i are normal subgroups of $\pi_1(\Sigma_{g_i})$. According to Lemma 3.5, we have an isomorphism $\gamma'^{-1}(N)/(I_1 \times I_2) \cong \pi_1(\Sigma_{g_1})/I_1 \cong \pi_1(\Sigma_{g_2})/I_2 \cong (\pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}))/\gamma'^{-1}(N) \cong \mathbb{Z}$. If $\gamma' = (\gamma_1, \gamma_2) \in \gamma'^{-1}(N)$ projects to a generator of $\gamma'^{-1}(N)/(I_1 \times I_2)$, then $\gamma' = (\gamma_1, 1)$ projects to a generator of $(\pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}))/\gamma'^{-1}(N)$. Clearly $\gamma'_*([\gamma', \gamma']) \notin N$ and its image generates $\pi_1(M_2)/N = \mathbb{Z}$. Also, $\gamma'_*([\gamma', \gamma'])$ projects to the generator of $\pi_1(\Sigma_{g_i})/I_i \cong \mathbb{Z}$ (see the proof of Lemma 3.5).

We claim that $\gamma'_*([\gamma', \gamma']) \neq 1$. Otherwise, for any $\gamma \in \pi_1(\Sigma_{g_1})$, $\gamma'_*([\gamma, 1]([\gamma^{-1}, 1])) = \gamma'_*([\gamma, \gamma^{-1}], \gamma_2) = 1$. Multiplying by γ_2^{-1} we get $[\gamma_1, \pi_1(\Sigma_{g_1})] \subset \ker(\gamma'_*)$. Analogously, $[\gamma_2, \pi_1(\Sigma_{g_2})] \subset \ker(\gamma'_*)$. $\gamma'_*([\gamma', \gamma']) = 0$ also implies that γ'_* sends $I_1 \times I_2$ surjectively onto N . For any $\gamma = (\gamma_1, \gamma_2) \in I_1 \times I_2$, $\gamma'^{-1}(\gamma) = (\gamma_1, \gamma_2)$ where $\gamma_1 \in \ker(\gamma'_*)$, but $\gamma'_*([\gamma', \gamma']) \gamma'_*([\gamma', \gamma'])^{-1}$ is not always equal to $\gamma'_*([\gamma', \gamma'])$, because $\pi_1(M_2) = N \rtimes_A \mathbb{Z}$ and A is not trivial. This leads to a contradiction.

Now since $\gamma'_*([\gamma', \gamma']) \neq 1$, $\ker(\gamma'_*) \subset I_1 \times I_2$. We wish to determine the quotient group $(I_1 \times I_2)/\ker(\gamma'_*)$. This is a normal subgroup of $\gamma'^{-1}(N)/\ker(\gamma'_*) \cong N$ with quotient $\gamma'^{-1}(N)/(I_1 \times I_2) \cong \mathbb{Z}$. Depending on the geometry of M_2 , N is isomorphic to \mathbb{Z}^3 or $\mathbb{Z}^2 \rtimes_C \mathbb{Z}$ (C is idempotent). Anyway, $(I_1 \times I_2)/\ker(\gamma'_*)$ must be isomorphic to \mathbb{Z}^2 .

Denote by J_i the intersection $\ker(\gamma'_*) \cap I_i$, and by P_i the image of the projection: $\ker(\gamma'_*) \hookrightarrow I_1 \times I_2 \rightarrow I_i$. For any $\gamma \in P_1$ and $\gamma' \in \pi_1(\Sigma_{g_1})$, by definition, there exists $\gamma'' \in I_2$ such that $\gamma'_*([\gamma, \gamma'']) = 1$. Also $1 = \gamma'_*([\gamma, \gamma']([\gamma', \gamma''])^{-1}) = \gamma'_*([\gamma, \gamma']([\gamma', \gamma''])^{-1})$. Multiplying by $(\gamma_2^{-1}, \gamma_2^{-1}) \in \ker(\gamma'_*)$, we have the conclusion $[P_1, \pi_1(\Sigma_{g_1})] \subset \ker(\gamma'_*)$. The conclusion for P_2 is analogous.

By Lemma 3.5, $\ker(\gamma'_*)/(J_1 \times J_2) \cong P_1/J_1 \cong P_2/J_2 \cong (P_1 \times P_2)/\ker(\gamma'_*)$. The last

group is a subgroup of $(I_1 \times I_2)/\ker(\ast) \cong \mathbb{Z}^2$. Hence there are three cases:

(a) $(P_1 \times P_2)/\ker(\ast) = 0$. This means that $\ker(\ast) = P_1 \times P_2$ and $(I_1/P_1) \times (I_2/P_2) \cong \mathbb{Z}^2$. Hence either $I_1/P_1 \cong I_2/P_2 \cong \mathbb{Z}$, or $I_1 = P_1, I_2/P_2 \cong \mathbb{Z}^2$.

For the former case, $\pi_1(\Sigma_{g_1})/P_1$ must be isomorphic to \mathbb{Z}^2 or the Klein bottle group since there is an exact sequence

$$0 \rightarrow \mathbb{Z}(= I_1/P_1) \rightarrow \pi_1(\Sigma_{g_1})/P_1 \rightarrow \mathbb{Z}(= \pi_1(\Sigma_{g_1})/I_1) \rightarrow 0$$

So does $\pi_2(\Sigma_{g_1})/P_2$. If both are \mathbb{Z}^2 , then \ast factors through $\mathbb{Z}^2 \times \mathbb{Z}^2$ which is abelian, and cannot have finite-index image. If any one $\pi_1(\Sigma_{g_i})/P_i$ is the Klein bottle group, then we take a double cover of Σ_{g_i} and can derive the same contradiction.

For the latter case, $\pi_1(\Sigma_{g_1})/P_1 = \pi_1(\Sigma_{g_1})/I_1 \cong \mathbb{Z}$. By Lemma 6.2, \ast is homotopic to a map $\ast' : \Sigma_{g_1} \times \Sigma_{g_2} \rightarrow S^1 \times \Sigma_{g_2} \rightarrow M_2$ which has zero degree for dimension reason.

(b) $(P_1 \times P_2)/\ker(\ast) \cong \mathbb{Z}$. This means that $(I_1/P_1) \times (I_2/P_2) = (I_1 \times I_2)/(P_1 \times P_2) \cong \mathbb{Z}$, which implies that one of the groups I_i/P_i , say I_1/P_1 , is trivial. As explained above, $\mathbb{Z} \cong (P_1 \times P_2)/\ker(\ast) \cong P_1/J_1$, so $I_1/J_1 \cong \mathbb{Z}$. Again we have $\pi_1(\Sigma_{g_1})/J_1 \cong \mathbb{Z}^2$ (or the Klein bottle group; and we take double cover). By Lemma 6.2, \ast is homotopic to a map $\ast' : \Sigma_{g_1} \times \Sigma_{g_2} \rightarrow T^2 \times \Sigma_{g_2} \rightarrow M_2$. Denote by \ast'' the latter map. For $S^1 \times \mathbb{E}, N^4, S^4_{m,n}, S^4_0, S^4_1$ geometry, the center of $\pi_1(M_2)$ is either trivial or isomorphic to \mathbb{Z} . Hence after a homotopy, \ast'' compresses one S^1 factor of T^2 , therefore has zero degree for dimension reason. For $N^4 \times \mathbb{E}$ geometry, Proposition 7.3 and Proposition 2.3 tells us that $\deg(\ast'') = 0$.

(c) $(P_1 \times P_2)/\ker(\ast) \cong \mathbb{Z}^2$. This means that $P_1 = I_1$ and $P_2 = I_2$. From the conclusion above, $[I_i, \pi_1(\Sigma_{g_i})] \subset \ker(\ast)$ ($i = 1, 2$). As a smaller subgroup, $[I_i, \mathbb{Z}_i] \subset \ker(\ast)$. Now we need another simple lemma:

Lemma 9.1 *Let G be a group, H_1, H_2, H_3 be subgroups. Assume that H_2 is normal. Then we have $[H_1, H_2 H_3] \subset [H_1, H_2] N([H_1, H_3])$. The notation $N(H)$ denotes the smallest normal subgroup of G containing H (abuse of notation).*

Proof: It is easy to check that for any $i \in H_i$, $[i, [j, k]] = [i, j] [i, k]^{-1}$. \square

Proposition 9.1 cont'd: It is easy to see that $[\pi_1(\Sigma_{g_i}), \mathbb{Z}_i] = [I_i \mathbb{Z}_i, \mathbb{Z}_i] = [I_i, \mathbb{Z}_i] \subset \ker(\ast)$. Recall that I_i is normal in $\pi_1(\Sigma_{g_i})$. Use Lemma 9.1, we have $[\pi_1(\Sigma_{g_i}), \pi_1(\Sigma_{g_i})] = [\pi_1(\Sigma_{g_i}), I_i \mathbb{Z}_i] \subset \ker(\ast)$ ($\ker(\ast)$ is normal). Thus \ast factors through a map $\pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}) \rightarrow \mathbb{Z}^{2g_1+2g_2} \rightarrow \pi_1(M_2)$ which cannot be surjective. We get a contradiction. \square

Corollary 9.2 $\Sigma_{g_1} \times \Sigma_{g_2}$ does not dominate any $N^4 \times \mathbb{E}$ -manifold.

Proof: Any finite cover of $\Sigma_{g_1} \times \Sigma_{g_2}$ is finitely covered by $\Sigma_{g'_1} \times \Sigma_{g'_2}$ for some g'_1, g'_2 . \square

The dominating ability of irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ is much weaker:

Proposition 9.3 *Let M be a irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ -manifold. Then M dominates $S^2 \times S^2$.*

Proof: According to Matsushima's result in [8], M satisfies $h^2(M) = 2 + h^{(2,0)}(M)$, where $h^{(2,0)}$ is the dimension of $H^{(2,0)}(M)$ (with respect to the natural Kahler structure on M). Simple calculation of the dimension of cohomology groups leads to the conclusion $\sigma(M) = 0$ (Wall has an elegant explanation of this in [17, §6]). It is well-known that any indefinite unimodular quadratic form on an integral lattice represents zero (see [11, §V.2]). Hence M satisfies the conditions of Proposition 5.1. \square

Proposition 9.4 *Let M be a irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ -manifold. Then M does not dominate any geometric manifold except $\mathbb{C}\mathbb{P}^2$, $S^2 \times S^2$ and its quotients.*

Proof: The Margulis Normal subgroup Theorem (see [7, Chapter IV]) implies that any normal subgroup of $\pi_1(M)$ has finite index.

Let M' be a typical covering manifold of any geometry except $\mathbb{C}\mathbb{P}^2$ and $S^2 \times S^2$; we have $\chi_1(M') > 0$ (check the geometries one by one). By Margulis's Theorem, M cannot dominate M' , otherwise the inverse image of $[\pi_1(M'), \pi_1(M')]$ will be an infinite-index normal subgroup of $\pi_1(M)$. \square

In fact, all domination relations between irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ -manifolds are trivial, i.e. M dominates N if M covers N . This follows from Margulis's Theorem and Mostow Rigidity.

One can easily see that no geometry dominates $\mathbb{H}^2 \times \mathbb{H}^2$. The other geometries are either solvable, or has S^2 or S^3 factors, or has non-trivial center. The $\mathbb{H}^2 \times \mathbb{H}^2$ -part of Diagram 1-1 is thus complete.

Completion of proof of Theorem 7.5: Except for what we have proved in Proposition 7.4, we only need to prove: When $X = \mathbb{H}^2 \times \mathbb{H}^2$ and Y is non-contractible, $X \twoheadrightarrow Y$ implies $X \times \mathbb{E}^n \twoheadrightarrow Y \times \mathbb{E}^n$.

It suffices to prove $M \times T^n$ does not dominate $S^3 \times T^{n+1}$, where M is any irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ -manifold. For the same reason as in Proposition 9.4, we can prove that $M \times T^n$ dominates another manifold M_1 only if $\chi_1(M_1) \leq 0$. \square

Remark: We can regard the reducible and irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ as two different geometries. In this case, the reducible one dominates $\mathbb{H}^2 \times \mathbb{E}^2$ and the irreducible one dominates $S^2 \times S^2$. If we regard $\mathbb{H}^2 \times \mathbb{H}^2$ as one geometry, it can only dominate $S^2 \times S^2$. Proposition 7.5 holds for both cases.

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