DELOCALIZATION OF SCHRÖDINGER EIGENFUNCTIONS

Abstract.

1. Some hi o

“on th er hand, clas- sical stic ... tegrable system ... for ... ne as ... verage”.

$qp - pq =$

$h/2$.

H

A $i\hbar \frac{dA}{dt} =$ H, $[\cdot, \cdot]$

$E_n - E_m$, (E_n) H.

wave mechanics

m

V 100

$\frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi$

Δ $= (i,$

$x \in \mathbb{R}^3$ *wave function*

(1)

$H = -\frac{\hbar^2}{2m}\Delta + V$.

$L^2(\mathbb{R}^3)$ ϕ_n $E_n \in \mathbb{R}$.

(1)

$) = \sum_n c_n$

$c_n \in$ $= 0$.

$E_n - E_m$

101

t

t

1 2

H ($H\phi = E\phi$, $E \in \mathbb{R}$),

$(x;t)$ (1).

ensembles). (2).

.1

2. High f e enc delo ali a ion

(M, g) d,
Δ M.
$L^2(M; \mathrm{Vol})$, Δ
$0 <$ $\leq$ $\longrightarrow +\infty$,
$(\phi_k)_{k\in\mathbb{N}}$ $L^2(M;$ $)$,

$$\phi_k = - \ _k\phi_k.$$

M ,
(. ϕ_k ∂M ,
$\mathbb{R}$,
fi ,
ffi ,
fi .

$_k \longrightarrow +\infty$.
semi lassical l t

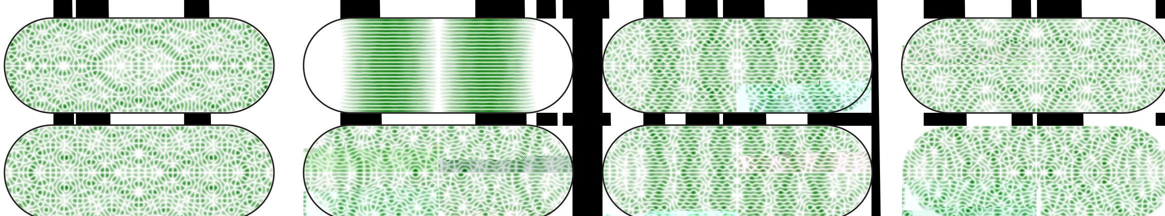

Fig e 1. $|\phi_n(x;$ $= 3 9$.

2.1. T the geod ic flow . Δ $_k = -$
$\hbar^2\Delta\phi = E\phi$ $_k = E$
) (V ( ) 3 2 30/ 11. 2 1

" ".

100 .

$-\hbar^2\Delta$ M

, M ,

T^*M *cotangent bundle*

M $(x;\ )\in T^*M$ $x\in M$

(" " $\in T_x^*M$ (" ") $(x;\ )\in T^*M$,

$t\in\mathbb{R}$, $g^t(x;\ )\in T^*M$,

$(g^t)_{t\in\mathbb{R}} : T^*M \longrightarrow T^*M$ g^{t+s} g^t g^s g^0

unit cotangent bundle $S^*M = \{(x;\ )\in T^*M; \|\ \|_x = 1\}$

g^t.

, ($_k\longrightarrow+\infty$),

$= i\Delta$

wave fronts

$^\epsilon(x$

(2), ($\epsilon \sim {}_k^{-1/2}$). $e^{t\Delta}$, t (2) ( ) t (2) .

2. L^p-norms as measures of delocalization ? 1 , ϕ_λ $-\Delta\phi_\lambda =$, $\|\phi_\lambda\|_{L^2} = 1$ L^∞-

Theorem 1. *(known as Hörmander's bound)*

$$\|\phi_\lambda\|_\infty = O(\lambda^{(d-1)/4}).$$

, . L^p , $\leq p$ $+\infty$

Theorem 2 (10).

$$\|\phi_\lambda\|_{L^p} = O(\lambda^{\frac{\mu(p)}{2}})$$

where

- $(p) =$ $\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}$ *for* $\frac{2(d+1)}{(d-1)} \leq p \leq +\infty$;
- $(p) = \frac{d-1}{2}$ $\left(\frac{1}{2} - \frac{1}{p}\right)$ *for* $2 \leq p \leq \frac{2(d+1)}{(d-1)}$.

M. d M $p_c = \frac{2(d+1)}{(d-1)}$. $\mathbb{S}^d$ $p \geq p_c$ ( ), $p \leq p_c$ ( ). , L^∞ . $p >$ L^p- ", $x \in M$, $\mathcal{L}_x$ S_x^*M x, . .

$$\mathcal{L}_x = \{ \in S_x^*M, \exists t > 0; g^t(x;\nu) \in S_x^*M\}$$

x S_x^*M.

Theorem 3 (, 10). • *Assume there exists a subsequence* ${}_{n_k} \longrightarrow +\infty$ *and* $C > 0$ *such that* $\|\phi\|_\infty \geq C$ $\lambda^{\frac{\mu(\infty)}{2}}$. *Then there exists such that* ${}_x(\mathcal{L}_x) > 0$;

- *If M is real analytic, the existence of such subsequence* λ_{n_k} *is to the existence of* x *such that* ${}_x = S_x^*M$, *and the first return map* ${}_x : S_x^*M \longrightarrow S_x^*M$ *possesses an absolutely continuous invariant probability measure. (Moreover, in* *that case, there exists* $t_0 > 0$ *such that* $g^{t_0}(x;v) \in S^*_x M$ *for all* $v \in S^*_x M$*, that is, there is a common return time).*

- *If* M *is real analytic and* $\dim M = 2$*, the existence of such subsequence* $\phi_{\lambda_{n_k}}$ *is* … *to the existence of* $x \in M$ *and* $t_0 > 0$ *such that* $g^{t_0}(x;v) = (x;v)$ *for all* $v \in S^*_x M$*.*

… *Anosov property,* … 3

$$\|\phi_\lambda\|_{L^p} = o\left(\lambda^{\frac{\mu(\infty)}{2}} \right) \; (\ldots 1 \ldots o).$$

Theorem 4. *(i) (* … 1 … *) If* $d = 2$ *and* M *has no conjugate points, or if* $d \geq 2$ *and* M *has non-positive sectional curvature, for* $p = +\infty$*,*

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\lambda^{\frac{\mu(p)}{2}}}{\sqrt{\log \lambda}} \right):$$

(i') (… 30 *) Statement (i) also holds if* M *has no conjugate points for all* $d \geq 2$*.*

(ii) (… 3 *) (i) holds for all* $p > p_c$*.*

(iii) (… 2, 22 *) If* M *has non-positive sectional curvature, for* $p < p_c$ *there exists* $\sigma(p;d) > 0$ *such that*

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\lambda^{\frac{\mu(p)}{2}}}{(\log \lambda)^{\sigma(p,d)}} \right)$$

(iv) (… 23 *) Statement (iii) still holds for* $p = p_c$*.*

… 1 …

3. The Shnirelman theorem and the Quantum Unique ergodicity conjecture.

… $|\phi_k(x)|^2 d\mathrm{Vol}(x)$.

… $k \longrightarrow +\infty$) … $\phi_k(x)|^2 d\mathrm{Vol}(x)$

“ ” ([redacted]). [redacted] fi [redacted] fl ,

, [redacted] $a : S^*M \longrightarrow \mathbb{R}$, [redacted] $(x_0;\ _0) \in$ [redacted] $^*M$, [redacted] $\frac{1}{T}\int_0^T a \circ g^t(x_0;\ _0)dt$ [redacted] $T \longrightarrow +\infty$ [redacted] $\int_{S^*M}$ [redacted] dL [redacted] L [redacted] S^*M (. . [redacted]), [redacted] T [redacted] M [redacted]

Quant[redacted] **dicity T**[redacted] **elman** [redacted] **heorem).**

Theorem [redacted] , [redacted] 10 , 0, 11 [redacted] *Let* ([redacted] *be a compact* [redacted] *with* [redacted] *ric norm* [redacted] *lized so that* Vo[redacted]$M) =$ [redacted] *all* Δ *the Lapla*[redacted] *operat*[redacted] M [redacted] *e that th*[redacted] *geodesic flow of* [redacted] *is* [redacted] *with respect to* [redacted] *be an or*[redacted]*honormal basis o*[redacted]$^2(M;$ [redacted] *de of eigenfunct*[redacted]

$$\Delta [redacted] = - \ _k [redacted] \ _k \le \ _{k} [redacted] \longrightarrow +\infty:$$

Let a *be* [redacted] *funct*[redacted] M [redacted]

(3) [redacted] $\displaystyle \sum_{k,\lambda_k \le}$ [redacted] $\langle$ [redacted] $; a\phi$ [redacted] $\displaystyle \int_M a(x$ [redacted] $d\,\mathrm{Vol}(x)\Big|^2 \underset{\lambda\longrightarrow+\infty}{\longrightarrow}$ [redacted]

where the [redacted] *facto*[redacted] k [redacted] $_k \le \ \}|$ [redacted]

$\langle$ [redacted] $;$ [redacted] $_{L^2(M)}$ [redacted] $a(x$ [redacted] $) d\,\mathrm{Vol}(x)$

Remark [redacted] *at there* [redacted] *xists a subset* S [redacted] $\mathbb{N}$ [redacted] *ity* 1 *such that*

( ) [redacted] $\langle$ [redacted] $; a\phi_k\rangle_{n}$ [redacted] $_{,n\in S}\displaystyle\int_M a(x$ [redacted] $d\,\mathrm{Vol}(x):$

In addition, [redacted] *e fa*[redacted] *that the* [redacted] *e of contin*[redacted] *ous functions is* [redacted] *parable* [redacted] *one can actually find* $S \subset$ [redacted] *f density* 1 *such* [redacted] ( ) *holds* [redacted] $a \in C^0(M$ [redacted] *In oth*[redacted] *words, the sequence of m*[redacted] *res* $(|\phi_k(x)|^2 d\,\mathrm{V}$ [redacted] $)_{n\in S}$ *converges weakly to th*[redacted] *uniform* [redacted] *measure* $d\,\mathrm{Vol}(x)$.

Actually, the full [redacted] *atement of the theorem says that there exists a subset* $S \subset \mathbb{N}$ *of density* 1 *such that*

( ) $$\langle\phi_k; A\phi_k\rangle \underset{n\longrightarrow+\infty, n\in S}{\longrightarrow} \int_{S^*M} \ ^0(A) dL$$

for every pseudodifferential operator A *of order* 0 *on* M. *On the right-hand side,* $\ ^0(A)$ *is the principal symbol of* A, *that is a function on the unit cotangent bundle* S^*M. *Equation 4 corresponds to the case where* A *is the operator of multiplication by the function* a.

, 11 .

1,

$\hbar \longrightarrow$

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(3) a

1,

Qu ntu niq Ergodi co ecture.

M

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rva res

, 1

20

2 . E tro y and s pp o e icl sical easu es.

$\langle \ A \ \rangle$ A

$(\phi_{n} \ A \phi_{n})$

$(A$,

“ ” ϕ_{n_k}.

(ϕ_{n_k}).

, $g^t_\sharp$ = $t \in \mathbb{R}$.

$\longrightarrow +\infty$,

2.1.

Theorem 7. 3 *Assum M is a compact Riemannian manifold with negative sectional curvature. Assume $\langle \phi_{n_k}, A\phi_{n_k}\rangle$ converges to $\int_{S^*M}$ $^0(A)d$ for all A. Then has positive entropy.*

$h_{KS}(\ )$,

- $_{KS}(\ ) = 0$
- $\mapsto h_{K}$ $h_{KS}(\ \ _1 + 1 -\ )$ $h_{KS}(\ _1) + (\ -\ )h_{K}$ $_1$, $_2$, $\in$ $;1$
- (-M

$$h_{KS}(\ ) \le \int \left(\sum_{j=} \ \right) d(\ )$$

$^+_j(\ )$

- L $2)$.
- d ($2d-1$), $d-1$

$$h_{KS} \le d$$

Corollary 1. *$\in S^*M$ be the union all points lying a periodic trajectory on the geodesic flow all M has negative curvature countably many periodic geodesics). Let as in Theorem 7. Then ( ) <*

scar (,).

strong scarred

[redacted] *partial scar*, [redacted] $(\Gamma) > 0$ [redacted]

Cor[redacted]**lary 2.** *The su*[redacted]*port of* [redacted] *as Hausdorff dimension* $>$ [redacted]

[redacted] ≥ 1 [redacted]

The[redacted]**em 8** [redacted]**).** *Assume* [redacted] *is* [redacted] *compact Riemannian man*[redacted]*old of di*[redacted]*ension* [redacted] *with co*[redacted]*sectio*[redacted]*al curvature* $-$[redacted] *has entropy greater than* [redacted]

[redacted] $d-1$ [redacted]

1=2. [redacted] $\frac{d-1}{2}$

Cor[redacted]**ary 3.** [redacted] *dim*[redacted]*nsion* $\geq$ [redacted]

[redacted] T^*M [redacted] d, [redacted]

Cor[redacted]**4.** [redacted] *be* [redacted] *ion o*[redacted]*all po*[redacted]*nts ly*[redacted]*a c*[redacted]*sed traje*[redacted]*ory on the geod*[redacted]*ic* [redacted]*low.* [redacted] *in T*[redacted] *7,* [redacted]*th M* [redacted]*of co*[redacted]*nega*[redacted]*ive curva*[redacted]*ure. Then* [redacted]

[redacted] h_{KS} [redacted] $\geq \frac{d-1}{2}$ [redacted] $d-1$, [redacted] $\leq 1=2$.

[redacted] $\sum_{j=1}^{d-1}$ [redacted]

[redacted] 1=2 [redacted]

Th[redacted]**eor**[redacted]**).** [redacted] $(\Omega) > 0$ *for any* [redacted]*on-empty ope*[redacted] [redacted]

2. . Some questions on non-compact manifolds.

Absolutely continuous spectrum.

Large … localisation … non-compact manifolds.

regular ,

3.

3.1. Overview of the problem. $G = (V, E)$.

localized, *delocalized*

$\subset \mathbb{R}$,

- *spectral localization :* I,
- *exponential localization:*
- *dynamical localization* I

- *spectral delocalization* I,
- *ballistic transport*

spatial delocalization.

$N \to \infty$

N, $(\phi_j)_{j=1}^N$

ϕ_j

$\sum |\phi_j|$

- *ℓ^∞ norms* $|$, $\|\phi$ $1/\sqrt{N}$
- *ℓ^p norms:* ϕ_j $^{1/p - 1/2}$ ϕ
  non-ergodic (*multifractal*) $f(p) \neq 1/p - 1/2$.
- *Scarring :* $\sum_{x \in \Lambda}$ $\geq 1 -$
  $(\,)_{x\in\Lambda}$ ^{2} ϵ Λ
- *Quantum ergodicity :* $\{ \ldots \}$,
  $\sum |\phi_j(x)|^2$ $a($

most j. *all j*, *quantum unique ergodicity.*

almost sure ℓ^∞-

random *random*

2, , 3 1 1 , 13, 1 .1. ff

3. 3

$(q+1)$-

$(q+1)$

$q+1$ $(G_N)_{N\in\mathbb{N}}$ $(V_N; E_N)$

fi V_N

$$\mathcal{A}_N f(x) = \sum_{x\sim y} \ldots$$

x x y

$$\ldots f(x) = \sum f(y) \ldots (x)) :$$

$$\mathcal{A}_N \ldots I = \ldots$$

Theorem $(G_N$ *a sequence of* $(q+1)$*-regular graphs* *with finite* $G_N = (V_N; E_N)$ V_N $\{1; \ldots; N\}$ *Assume that*[1] *there exists* $c > 0; > 0$ *such that for any* $k \le c \ldots N$, *for any pair of vertices* $x, y \in V_N$,

$$|\{\text{paths of length } k \text{ in } G \text{ from } x \text{ to } y\}| \le q^{k\left(\frac{1-\delta}{2}\right)}$$

Fix $\epsilon > 0$. *Then, if* ϕ *is an eigenfunction of the discrete Laplacian on* G_N *and* $\Lambda \subset V_N$ *a set such that*

$$\sum_{x\in\Lambda} |\phi(x)|^2 \ge \epsilon \sum_{x\in V_N} |\phi(x)|^2;$$

then $|\Lambda| \ge N^{\alpha}$ *— where* $\alpha > 0$ *is given as an explicit function of* $\epsilon;$ *and* c.

$$H_N(\phi) = -\frac{1}{\log N}\sum_{x} |\phi(x)|^2 \ldots |\phi \ldots|^2$$

[1] $\ge \ln N$ 90

L^∞- ... $\|\phi\|_\infty = O((\log N)^{-1/}$ ...).

... =4 ...

(11) ...

3.3. **QE on ... egular grap...** ... , ... " ... "

... fi ... , ...

... $G_N = (V_N; E_N)$ ... 1)- ... ,

(EXP) ... . ,

> 0 ... $(q+1)^{-1}\mathcal{A}_N$ ... $\ell^2(V_N$ ... $1\} \cup [-1+ ; 1-$]

N.

... , ... -

... G_N ... - ... .

... $\mathcal{A}_N$ ...

0, ... (... 3).

(E ...) ... R,

$$\frac{|\{x \in V_N; \dots R\}|}{\dots} \dots 0$$

... " ... , ... r

... .

(E ...) ... ,

... 1 ... $(q+$ ...)- ...

... fi ... $(q+1$

... **(E ...)** ... ,

... $($ ... $^{)}; \dots;$ ... $)$ ... G_N ...

... $\subset \mathbb{R}$ ...

$$\dots |\{j; \dots \in \dots\}| \underset{N \to +\infty}{\longrightarrow} \int \dots \mathrm{m}(\ )d$$

... m ...)

o ... $\mathfrak{X}$).

(12) $$\int \ {}^{k}\mathrm{m}(\ )d \dots = \langle \ {}_o; \mathcal{A}^k_{\mathfrak{X}} \ {}_o \rangle_{\ell^2(\mathfrak{X})}$$

... $\mathfrak{X}$ ... $\mathfrak{X}$ ... o

... o ... x ... . m ... ,

... $(-2\sqrt{q}; 2\sqrt{q}$ ...

... $^{(N)}_{j}$ ... $(-2\sqrt{q}; 2\sqrt{q})$, ...

...); q ...

11.

Theorem 11 (M). *Let $(G_N) = (V_N; E_N)$ be a sequence of $(q+1)$-regular graphs with $|V$*[redacted]*$| = N$. Assume that (G_N) satisfies* **(BST)** *and* **(EXP)**.

Let $(\phi_1^{(N)}; \ldots ; \phi_N^{(N)})$ be an [redacted]*thonormal basis of eigenfunctions of $\mathcal{A}_N$ in $\ell^2(V_N)$.*

Let $a_N : V_N \longrightarrow \mathbb{C}$ be a seq[redacted]*ence of functions such that $\sup_N \sup_{x\in V_N} |a_N(x)| \leq 1$. Define $\langle a_N\rangle = \frac{1}{N}\sum_{x\in V_N} a_N(x)$.*

Then

$$\frac{1}{N}\sum_{j=1}^{N} [redacted] \phi_j^{(N)}; a_N\phi_j^{(N)}\rangle_{\ell^2(V_N)} - \langle a_N\rangle\Big|^2 \underset{N\longrightarrow+\infty}{\longrightarrow} 0.$$

, > 0

(13) $$\frac{1}{N}\Big|\Big\{j \in [1; N] [redacted] \langle\phi_j^{(N)}; a_N\phi_j^{(N)}\rangle_{\ell^2(V_N)} - \langle a_N\rangle\Big| > \Big\}\Big| \underset{N\longrightarrow+\infty}{\longrightarrow} 0.$$

$\langle\phi_j^{(N)}; a_N\phi_j^{(N)}\rangle_{\ell^2(V}$ $\phi_j^{(N)}$ $a_N\phi_j^{(N)}$,

$\sum_{x\in V_N} a_N(x)|\phi$[redacted]$(x)|^2$. 11

V_N,

$$\sum_{x\in V_N} |\phi_j^{(N)}(x)|^2 \quad \frac{1}{N}\sum_{x\in V_N} \quad (\quad)$$

a_N). (13)

N j,

3. . **No**[redacted]**-regular graph**[redacted] **from spectral to spatia**[redacted] **delocalization.**

regular .

,

quantu[redacted] *godicity theorem*

.

spectral d[redacted]

spat[redacted]*l delocalization* fi .

(

"If a l[redacted]*ge finite system* [redacted] *close (in the B*[redacted]*njamini-Schra*[redacted] *to an inf*[redacted]*ite sys-tem hav*[redacted]*g purely absolut*[redacted] *continuous spe*[redacted]*trum in an in*[redacted] *the eigen*[redacted]*ncti*[redacted]*ons (with eig*[redacted]*nvalues lying in* [redacted] *of the finite sy*[redacted]*tem satisfy q*[redacted]*city."*

$(G_N)_{N\in}$[redacted] 3

V[redacted] E G[redacted], $|V$[redacted] $=$

N N[redacted] $\longrightarrow \infty$. fi $\mathbb{C}$ $\mathbb{C}$

$$(\mathcal{A}_N f)(v) [redacted] \sum_{w\sim v} f(w)$$

v w v .

$\mathcal{A}_N$, (/ N [redacted] ∞.

(EXP) (G_N) [redacted] .

[redacted] , [redacted] fi [redacted] (

[redacted]) $P_N : \mathbb{C}^{V_N}$ [redacted] $\mathbb{C}^{V_N}$

[redacted])
$$(P_N f)(x) = \frac{1}{d_N(x)} \sum_{y \sim x} f(y)\,;$$

$d_N(x$ [redacted] x. **(EXP)** [redacted]

[redacted] 1 [redacted] P_N [redacted] ,

P_N [redacted] $[-1+$ [redacted] 1 [redacted] $] \cup \{1\}$, [redacted] > 0 [redacted] N.

[redacted] 1 [redacted] , [redacted] .

[redacted] G_N [redacted] .

[redacted] P [redacted]

([redacted] , 3).

[redacted] (G_N) [redacted]

(E[redacted]ST) [redacted] > 0,
$$\lim_{N [redacted] \infty} \frac{\{x \in V_N : \ _{G_N}(x) [redacted] r\}|}{N} = 0\,;$$

$_{G_N}($ [redacted] *injectivi[redacted] radius* x, [redacted] B_{G_N} [redacted])

.

[redacted] ([redacted]),

[redacted] (G_N),

[redacted] *ed graphs* [redacted] ,

[redacted] **(E[redacted]**

(E[redacted]SCT) [redacted] *tree* $[\mathcal{T};o]$ [redacted]

[redacted] $\{[\mathcal{T};o]\}$ [redacted] $\mathbb{P}$ [redacted]

([redacted] $\mathcal{A}_N$ [redacted] (V_N). [redacted] **SCT)** [redacted]

[redacted] $: \mathbb{R} \longrightarrow \mathbb{R}$,

(1 [redacted]) [redacted] $\sum_{j}$ [redacted] $\underset{N\longrightarrow+\infty}{\longrightarrow}$ [redacted] $_o$; [redacted] $(\mathcal{A}_\mathcal{T})$ [redacted] $_o\rangle =: \int$ [redacted] $dm(t)$,

$\mathcal{A}_\mathcal{T}$ [redacted] $\mathcal{T}$ [redacted] $(\mathcal{T})$.

[redacted] $\mathbb{P}$. [redacted] *ntegrated density of* [redacted] *s*

[redacted] .

[redacted] I [redacted]

([redacted] $\mathbb{P}$- [redacted]]). [redacted] $[\mathcal{T};o]$ [redacted]

$\mathbb{P}$). [redacted] x [redacted] $\in \mathbb{C} \setminus \mathbb{R}$ [redacted]

[redacted] $(x;y) =$ [redacted] $\mathcal{A}_\mathcal{T}$ [redacted] $)^{-1}$ [redacted] $_{\ell^2(\mathcal{T})}$ [redacted]

$v, w \in \mathcal{T}$ $v \sim w$, $\mathcal{T}^{(v|w)}$ $\mathcal{T}$ v w. $\mathcal{A}_{\mathcal{T}^{(v|w)}}$ $\mathcal{G}^{(v|w)}(\cdot;\cdot;\;)$ $_w^\gamma(v) = -\;^{(v|w)}(v;\;;\;)$.

(Green) I, $s > 0$

$$\sup_{\lambda \in I, \eta_0 \in (0,1)} \mathbb{E}\left(\sum_{y : y \sim o} |\operatorname{Im}\;_o^{\lambda + i\eta_0}(y)|^{-s}\right) < \infty.$$

(Green), fi ( ) $[\mathcal{T}; o]$

(1) $J = \langle\;; \mathbb{1}_J(\mathcal{A}_{\mathcal{T}})\;_o\rangle$ $J \subseteq \mathbb{R}$.

(Green) $\sup_{\lambda \in I, \eta_0 > 0} \mathbb{E}(|\mathcal{G}\;o; o)|^2) < \infty$.

$\mathbb{P}$- . . $[\mathcal{T}; o]$, $_o$ $\frac{1}{\pi}\operatorname{Im}\mathcal{G}^{+i0}(o; o)$. , **(Green** $\mathbb{P}$- . . $\mathcal{A}_{\mathcal{T}}$ I.

Green), < 0,

I **(Green)**, fi $\bar{I}_1$ I. G_N Γ_N $\widetilde{G_N}$, $\in \mathbb{C}$ $\widetilde{\mathcal{A}}_N$ $\widetilde{G_N}$

(1 $(\tilde{x}) = \;_{\tilde{x}}; (\;_N - \;)^{-1}\;\rangle_{\ell^2(\widetilde{G_N})}$.

Theorem 2 ( ). *Assume that* (G_N, W_N) *satisfies* ***(BST), (EXP)*** ***(Green)***

Call $(\;)_{j=1}^N$ *the eigenvalues of* $\mathcal{A}_N$ *on* $\ell^2(V_N)$, *and let* $(\phi_j^{(N)})_{j=1}^N$ *be a corresponding orthonormal eigenbasis.*

For each N, *let* $a = a_N$ *be a function on* V_N *with* $\sup_N \sup_{x \in V_N} |a_N(x)| \leq 1$.

Then

$$\lim_{\eta \to 0} \lim_{N \to \infty} \;\; \left|\sum_{x \in V_N} a(x)|\phi_j^{(N)}(x)|^2 - \sum_{x \in V_N} a(x)\;_{\lambda_j^{(N)} + i\eta}^N\right| = 0$$

for some family of probability measures on V_N, *indexed by a parameter* $\in \mathbb{C} \setminus \mathbb{R}$, *defined as follows*

$$(x) = \frac{\operatorname{Im}\tilde{g}_N^\gamma(\tilde{x}; \tilde{x})}{\sum_{y \in V_N} \operatorname{Im}\tilde{g}_N^\gamma(\tilde{y}; \tilde{y})}.$$

Here, $\tilde{x} \in \widetilde{G_N}$ is a lift of $x \in V_N$.

$\epsilon > 0$,

$$(1\ ) \quad \frac{1}{N}\left|\left\{ {}^{(N)}_{j} \in I_1 : \left|\sum_{x\in V_N} a(x)|\phi_j^{(N)}(x)|^2 - \sum_{x\in V_N} a(x)\ {}^{N}_{\lambda_j^{(N)}+i\eta_0}(x)\right| > \epsilon\right\}\right| \underset{N\to+\infty,\,\eta_0\downarrow 0}{\longrightarrow} 0 :$$

12 [redacted] $a_N = \mathbb{1}_{\Lambda_N}$,

$\Lambda_N \subset V$ [redacted] $\approx$ N [redacted] $\in (0;1)$. [redacted] (G_N) ,

$\widetilde{G_N}$ [redacted] N ([redacted] $(q+1)$- [redacted])

$\tilde{g}_N^{\gamma}(\tilde{x};\tilde{y})$ [redacted] $\mathcal{G}^{\gamma}(\tilde{x};\tilde{y})$ [redacted] .

$\mathcal{G}^{\gamma}(\tilde{x};\tilde{x}) = \mathcal{G}^{\gamma}(o;o)$ [redacted] $\tilde{x} \in \widetilde{G_N}$.

V [redacted]). [redacted] $\mathbb{1}_{\Lambda_N}\phi_j^{(N)}\|^2 \approx$ [redacted] $\phi_j^{(N)}$.

$\phi_j^{(N)}$ [redacted] $\Lambda_N \subset V_N$

[redacted] $|\phi_j^{(N)}|^2$.

[redacted] ${}^{N}_{\gamma}(\Lambda_N) =$ [redacted] N.

$c_\alpha > 0$ [redacted] $\Lambda_N \subset V_N$ [redacted] $|\Lambda_N| \geq$ [redacted],

$$(1\ ) \quad \liminf_{N\to\infty} \inf [redacted] {}_{+i\eta_0}(\Lambda_N) \geq 2c_\alpha :$$

[redacted] (1),

Co[redacted]**y 1**[redacted] *For any* [redacted] *1), there e*[redacted]*ts $c_\alpha > 0$ such tha*[redacted]*r any $\Lambda_N \subset V_N$ with $|\Lambda_N|$* [redacted] *N, we have*

$$\frac{1}{N} [redacted] \in I : \left\|\mathbb{1}_{\Lambda_N} [redacted] \right\|^2 < c_\alpha\right\} \underset{N\to+\infty}{\longrightarrow} [redacted]$$

[redacted] $\mathbb{1}_{\Lambda_N}\phi_j^{(N)}\|^2 \approx$ [redacted] $\phi_j^{(N)}$,

[redacted] $\phi_j^{(N)}\|^2 \geq$ [redacted] > 0 [redacted] $\phi_j^{(N)}$

[redacted] $F : \mathbb{R} \to \mathbb{R}$, [redacted] $\in I$,

$$(2\ ) \quad [redacted] \sum_{x\in V_N} F [redacted] (x)\Big) \underset{N\to+\infty}{\longrightarrow} [redacted] \left(F\left(\frac{\operatorname{Im}\mathcal{G}^{\lambda+i\eta} [redacted]}{\mathbb{E}(\operatorname{Im}\mathcal{G}^{\lambda+i} [redacted]))}\right)\right) :$$

[redacted] $N\ {}^{N}_{\lambda+i\eta_0}(x))$ (

V_N [redacted] $\left(\frac{\operatorname{Im}\mathcal{G}^{\gamma}(o,o)}{\mathbb{E}(\operatorname{Im}\mathcal{G}^{\gamma}(o,o))}\right)$.

(x) [redacted] $1{=}N$ [redacted] N,

Re[redacted]**ark 14.** *The results prove*[redacted] *in* [redacted] *actually hold for more ge*[redacted] *Schrödinger operators th*[redacted] *adjacency matrices* [redacted] *one can conside*[redacted] *weighte*[redacted] *Laplaci*[redacted] *the conductances on th*[redacted]*ges) and add a potential; in other words, on each G_N, we* [redacted] *consider a discrete Sc*[redacted]*dinger operator $\mathcal{H}_N$.* [redacted] *The limiting object in assumption* **(BSCT)** *is now a random* *rooted tree $[\mathcal{T}; o]$ endowed with a random Schrödinger operator $\mathcal{H}$. Assumption **(Green)** has to be modified, replacing the adjacency matrix $\mathcal{A}$ by the operator $\mathcal{H}$. Similarly, in the statement of the theorem, the Green functions $\tilde{g}_N^{\gamma}$ to be considered are those of $\mathcal{H}_N$ lifted to the universal cover $\widetilde{G_N}$.*

Remark 15. *In particular, our result applies to the case where the limiting system $([\mathcal{T}; o]; \mathcal{H})$ is $\mathcal{T} = \mathfrak{X}$ (the $(q+1)$-regular tree) with an arbitrary origin o, and $\mathcal{H} = \mathcal{H}_\epsilon = \mathcal{A} + \epsilon\mathcal{W}$ where $\mathcal{W}$ is a random real-valued potential on $\mathfrak{X}$. More precisely the values $\mathcal{W}(x)$ $(x \in \mathfrak{X})$ are i.i.d. random variables of common law . This is known as the Anderson model on $\mathfrak{X}$. It was shown by A. Klein that the spectrum of $\mathcal{H}_\epsilon$ is a.s. purely absolutely continuous on $I = (-2\sqrt{q} + \ ; 2\sqrt{q} - \ )$, provided ϵ is small enough (depending on). This just assumes a second moment on . Under stronger regularity assumptions on , one can show that Assumption **(Green)** holds on I (see 11 , following Aizenman-Warzel 2). Examples of sequences of expander regular graphs G_N with discrete Schrödinger operators $\mathcal{H}_N$ converging to $([\mathfrak{X}; o]; \mathcal{H}_\epsilon)$ are given in 10 .*

Remark 16. *Examples of sequences of satisfying our three assumptions were investigated in 11 . In the examples considered there, the limiting trees $\mathcal{T}$ are fi ; roughly speaking, those are trees where the local geometry can only take a finite number of values. If $\mathcal{A}$ is the adjacency matrix of such a tree, we showed in 11 that the spectrum of $\mathcal{A}$ is a finite union of closed intervals, and that there are a finite number of points $y_1; \dots; y_\ell$ in such that Assumption **(Green)** holds on any I of the form $\setminus ([y_1 - \ ; y_1 + \ ] \cup \dots \cup [y_\ell - \ ; y_\ell + \ ])$ (for any > 0). We showed – extending Remark 15 – that on such trees, Assumption **(Green)** remains true after adding a small random potential to $\mathcal{A}_\mathcal{T}$. Finally, we showed the existence of sequences (G_N) converging to $\mathcal{T}$ and satisfying the **(EXP)** condition.*

. PE PEC -E AND LINK I H O HE O K

.1. **Random regular graphs.** 11

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cency mat $a_N : \{1; \dots; N$ $\mathbb{R}$, [redacted] $(q+1)$- $\{1;\dots;N\}$ $\sum_{x=1}^{N} a_N(x)|\phi_j^{(N)}(x)|^2$ $\langle a_N\rangle$ *all* j, [redacted]

Theorem 17 ([redacted] - 1). *Let ! be such tha* [redacted] $\sqrt{q} \geq (!$ [redacted] $1)2^{2\omega+45}$.

(i) With probability $\geq$ [redacted] $1 - o(N^{-\omega+8})$ *on the choice of the graph* [redacted]

$$\|\phi_j\|_\infty \leq \frac{(\log N)^{121}}{\sqrt{N}}$$

for all eigenfunctions a[redacted]*sociated to eigenvalues such that* $|\ _j \pm 2\sqrt{\ }| > (\log N)^{-3/2}$.

(ii) (Quantum Uniqu[redacted] *Ergodicity for random regular graphs)* [redacted]*ven an observable* a_N : $\{1;\dots;N\} \longrightarrow \mathbb{R}$, *we* [redacted]*ave, with probability* $\geq 1 - o(N^{-\omega+8})$ *on* [redacted] *he choice of the graph, for* N *large enough,*

$$\left|\sum_{x=1}^{N} a_N [redacted] x)|\phi_j^{(N)}(x)|^2 - \langle a_N\rangle\right| \leq \frac{(\log N)^{250}}{N}\sqrt{\sum_x | [redacted] _N(x)|^2}; \quad (21)$$

for all eigenfunctions as[redacted]*ociated to eigenvalues* $_j \in (-2\sqrt{q}+\epsilon; 2\sqrt{\ }-\epsilon)$ *(bulk* [redacted] *genvalues).*

In particular, if $a_N =$ [redacted] 1_{Λ_N} *where* $\Lambda_N \subset \{1;\dots;N\}$, *we find*

$$\left| [redacted] \sum_{\Lambda_N} |\phi_j^{(N)}(x)|^2 - \frac{|\Lambda_N|}{N}\right| \leq \frac{(\log N)^{250}}{N}\sqrt{|\Lambda_N [redacted]}$$

([redacted] > 8 [redacted] $q > 2^{128}$).

∞- [redacted] , [redacted] $|\Lambda_N$ [redacted] $> (\log$ [redacted] $)^{500}$. [redacted] , [redacted] 11, [redacted] ∞,

[redacted]

most [redacted] j [redacted] $|$ [redacted] $| >$ [redacted] $^{1/2}$ [redacted]), [redacted] , [redacted] a_N [redacted] 11 (1), a_N [redacted] a_N [redacted] ([redacted]), [redacted] 1 [redacted] , [redacted] *lmost sure* [redacted]

Rem[redacted]**rk 1**[redacted] [redacted]*ote that we e*[redacted]*phasize*[redacted] *Theore*[redacted] *17 f*[redacted]*m 1* [redacted] *be*[redacted]*e our* [redacted]*a*[redacted] *concern here* [redacted]*s the* [redacted]*el*[redacted]*alizatio*[redacted] *of e*[redacted]*en*[redacted]*n*[redacted]*ions. T*[redacted]*e ma*[redacted] *focus of* [redacted] *1* [redacted] *is ho*[redacted]*e*[redacted]*er on the unive*[redacted]*sality of* [redacted]*e local s*[redacted]*ectra*[redacted] *sta*[redacted]*st*[redacted]*s for ra*[redacted]*dom re*[redacted]*ular graph*[redacted] *This wo*[redacted]*l*[redacted] *deserve a separ*[redacted]*te pa*[redacted]*r.*

[redacted] 2 [redacted] , [redacted] G_N [redacted] N [redacted] , [redacted] $\phi_j^{(N)}$,

$\sqrt{N}\phi_j(x)$ x $\{1, \dots, N\}$ $\mathcal{N}(0, \ _j^2)$ $0 \leq \ _j \leq 1$. $R \geq 0$, x $\{1, \dots, N\}$ $(\sqrt{N}\phi_j(y))_{y, d(y,x) \leq R}$ x $B_{\mathfrak{X}}(o, R)$. $_j^2 \Phi_{\lambda_j}(d(x, y))$ Φ_{λ_j} $(q+1)$- $\mathfrak{X}$. $_j \neq 0$, $o(\ )$. 10 , N^{α} . 11, a_N , (1 , (21), a_N (N), (1) (2).

.2. **Fr gra h aplacians to Hec e op ators.**

11 , “ ”. $\mathbb{S}$ $\Delta_{\mathbb{S}^2}$, $\mathbb{S}^2$ $\mathbf{J}$ $= \frac{1}{i}$ $- x$. $\mathbf{Y}$ 2. A $\int_{S M}$ $S M$, 1 .

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$\Delta_{\mathbb{S}^2}$, " "

$\mathbb{S}^2$ $g_1, \ldots, g_k$ fi $SO(3)$,

$$T_k f(x) = \sum_{j=1}^{k} \left(f(g_j x) + f(g_j^{-1} x) \right)$$

$\Delta_{\mathbb{S}^2}$.

Theorem 19 (, M 3). *Assume that $g_1, \ldots, g_k$ generate a free subgroup of $SO(3)$.*

For each ℓ, let $(\psi_j^{(\ell)})_{j=1}^{2\ell+1}$ be an orthonormal family of eigenfunctions of $-\Delta_{\mathbb{S}^2}$ of eigenvalue $\ell(\ell+1)$, that are also eigenfunctions of T_k.

Then for any continuous function a on $\mathbb{S}^2$, we have

$$\frac{1}{2\ell+1} \sum_{j=1}^{2\ell+1} \left| \int_M a(x) |\psi_j^{(\ell)}(x)|^2 d\mathrm{Vol}(x) - \int_M a(x) d\mathrm{Vol}(x) \right|^2 \underset{\ell \longrightarrow \infty}{\longrightarrow} 0.$$

T_k ℓ

T_k ℓ^{-1}.

$2k$- $N \sim \ell^2$, $g_1, \ldots, g_k$

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11. .

Remark 20. *T_k is not a pseudodifferential operator, so the argument sketched above to show that the basis $(Y_\ell^m)_{\ell \geq 0, |m| \leq \ell}$ could not satisfy quantum ergodicity does not apply here.*

Remark 21. *We note that for very special choices of rotations – rotations that correspond to norm n elements in an order in a quaternion division algebra, the operators T_k are called Hecke operators. It has been conjectured by Böcherer, Sarnak, and Schulze-Pillot that such joint eigenfunctions satisfy the much stronger quantum unique ergodicity property. The conjecture is still open.*

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.3. **Quantum ergodicity o Rie ann su aces of igh ge us.** 1 12

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T eorem 22 (0). *(S_N) be a s quen of hyperb aces, w ose ge s (equivale y, vol me) g es to*

EXP) *Assume th first e envalu $_1$(I) of $-\Delta$ S is ounded a 0 as $N \longrightarrow \infty$*

BSH) *Assume the are f shor geod s; in o er words (S) co the B jamin Schramm se to e hyperbolic : for y > 0*

$$N \longrightarrow +\infty \quad \frac{V \{x \in \quad (x) < \}}{V(S_N)} = 0$$

w re (x means the ectiv radi s at x

ix an terval $I \subset$ 4; + .

et $(\phi_i^{(N)})$ be an or ormal basis f eige nctions the La acia on

et $a = a_N : S_N \longrightarrow$ be s th $|a(x)|$ for all $\in S_N$. hen

$$\lim_{N\longrightarrow+\infty} \frac{1}{Vol(S} \sum_{\lambda_i(N} \left| \int_{S_N})|\phi_i^{(N)}(|^2 d - (\right|^2 = 0$$

w re $\langle a\rangle = \frac{1}{Vol(S_N)} \int_{S_N} a(x)dx$

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