DELOCALIZATION OF SCHRÖDINGER EIGENFUNCTIONS

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fi “on the other hand, classical statistical mechanics is essentially only concerned with Type b) [i.e. non integrable systems], for in this case the microcanonical average is the same as the time average”.

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q $qp - pq = i\hbar I$, $\hbar$

$h/2$. H

H .

A $i\hbar\frac{dA}{dt} = [A, H]$, $[\cdot,\cdot]$

. (

) ff $E_n - E_m$, (E_n) H.

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wave mechanics , ,

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$$(1) \qquad i\hbar\frac{\partial}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V\right)$$

Δ $= (t, x)$ t

$x \in \mathbb{R}^3$, *wave function.*

ff (1) ff

$H = -\frac{\hbar^2}{2m}\Delta + V$. , , fi

$L^2(\mathbb{R}^3)$ ϕ_n $H\phi_n = E_n\phi_n$ $E_n \in \mathbb{R}$.

(1)

$$(t, x) = \sum_n c_n \phi_n(x) e^{-itE_n/\hbar}$$

ffi $c_n \in \mathbb{C}$ $t = 0$.

ff $E_n - E_m$.

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, $|(x, t)|^2$

, , fi x, t.

o , ,

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1 2 ,

H

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H

H ($H\phi = E\phi$, $E \in \mathbb{R}$),

$(x;t)$ (1).

$|\ |^2$

ensembles

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$\mathbb{R}^2$,). 43

chaotic,

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- 103 ,
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arithmetic

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2. High f e enc delocali a ion

(M, g) d,

Δ M.

$L^2(M, \mathrm{Vol})$, Vol . Δ

$_0 = 0 < \ _1 \leq \ _2 \leq \longrightarrow +\infty$,

$(\phi_k)_{k\in\mathbb{N}}$ $L^2(M, \mathrm{Vol})$,

$$\Delta\phi_k = - \ _k\phi_k.$$

M , , -

(. . ϕ_k ∂M). M ,

$\mathbb{R}^2$

fi ,

ffi

fi .

, fi

$_k \longrightarrow +\infty$.

semiclassical limit,

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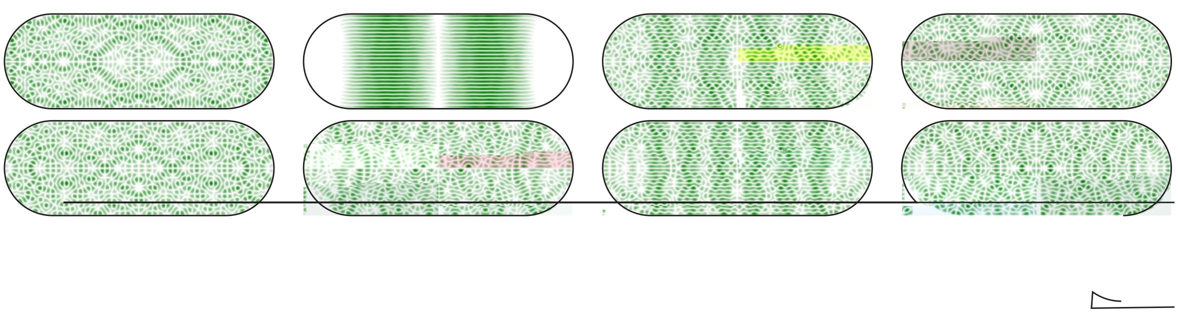

Fig e 1. $|\phi_n(x, y)|^2$ -

, $n = 319$.

. .

2.1. The role of the geodesic flow .

$\Delta\phi_k = - \ _k\phi_k$

$-\hbar^2\Delta\phi = E\phi$ ($_k = E\hbar^{-2}$)

(1) (V $_k$ ∞

“ ”.

ȯ 100 .

ȯ $-\hbar^2\Delta$ M

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, . M ,

, fl . T^*M *cotangent bundle*

M . $(x;\ ) \in T^*M$ $x \in M$

(“ ”) $\in T^*_x M$ (“ ”). $(x;\ ) \in T^*M$,

$t \in \mathbb{R}$, $g^t(x;\ ) \in T^*M$

t x .

$(g^t)_{t\in\mathbb{R}} : T^*M \longrightarrow T^*M$ fl ff , $g^{t+s} = g^t \circ g^s$ g^0

. fl .

, *unit cotangent bundle* $S^*M = \{(x;\ ) \in T^*M; \|\ \|_x = 1\}$

g^t.

, ($_k \longrightarrow +\infty$), ȯ

$$\frac{@}{@t} = i\Delta$$

wavefronts . ,

$^\epsilon(x$

ϕ (2),

($\epsilon \sim {}_{k}^{-1/2}$).

ffi ff

$e^{i\epsilon t\Delta}$, t

– (2) ( ) t

(2) .

2.2. L^p-norms as measures of delocalization ?

fi

1

ϕ_λ

$-\Delta\phi_\lambda = \phi_\lambda$, $\|\phi_\lambda\|_{L^2} = 1$. L^∞-

Theorem 1. *(known as Hörmander's bound)*

$$\|\phi_k\|_\infty = O({}_{k}^{(d-1)/4}).$$

, . L^p , $2 \leq p \leq +\infty$

Theorem 2 (, 10).

$$\|\phi_\lambda\|_{L^p} = O({}^{\frac{\mu(p)}{2}})$$

where

- $(p) = d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}$ *for* $\frac{2(d+1)}{(d-1)} \leq p \leq +\infty$*;*
- $(p) = \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$ *for* $2 \leq p \leq \frac{2(d+1)}{(d-1)}$.

M. d

M. $p_c = \frac{2(d+1)}{(d-1)}$.

$\mathbb{S}^d$ $p \geq p_c$ (

2) $p \leq p_c$ (

). , L^p- ,

. $p > p_c$,

, L^p-

" ",

$x \in M$, $\mathcal{L}_x \subset S_x^*M$ x, . .

$$\mathcal{L}_x = \{v \in S_x^*M; \exists t > 0; g^t(x; v) \in S_x^*M\}.$$

$_x$ S_x^*M.

Theorem 3 (- 10 , 10 , 110).

- *Assume there exists a subsequence* $_{n_k} \longrightarrow +\infty$ *and* $C > 0$ *such that* $\|\phi_{\lambda_{n_k}}\|_\infty \geq C {}^{\frac{\mu(\infty)}{2}}$. *Then there exists* x *such that* $_x(\mathcal{L}_x) > 0$*;*
- *If* M *is real analytic, the existence of such subsequence* $\phi_{\lambda_{n_k}}$ *is* *to the existence of* x *such that* $\mathcal{L}_x = S_x^*M$, *and the first return map* $_x : S_x^*M \longrightarrow S_x^*M$ *possesses an absolutely continuous invariant probability measure. (Moreover, in*

that case, there exists $t_0 > 0$ *such that* $g^{t_0}(x;v) \in S_x^*M$ *for all* $v \in S_x^*M$, *that is, there is a common return time).*

- *If* M *is real analytic and* $\dim M = 2$, *the existence of such subsequence* $\phi_{\lambda_{n_k}}$ *is* … *to the existence of* $x \in M$ *and* $t_0 > 0$ *such that* $g^{t_0}(x;v) = (x;v)$ *for all* $v \in S_x^*M$.

… *Anosov property,* … x … 3 … $\|\phi_\lambda\|_{L^\infty} = o\left(\frac{\mu(\infty)}{2}\right)$ (… O … 1 … o).

Theorem 4. *(i)* (… 1 … 1) *If* $d = 2$ *and* M *has no conjugate points, or if* $d \geq 2$ *and* M *has non-positive sectional curvature, for* $p = +\infty$,

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\frac{\mu(p)}{2}}{\sqrt{\log}}\right):$$

(i') (… 30) *Statement (i) actually holds if* M *has no conjugate points, for all* $d \geq 2$.

(ii) (… 3) *(i) holds for all* $p > p_c$.

(iii) (… 24, 22) *If* M *has non-positive sectional curvature, for* $p < p_c$, *there exists* $(p;d) > 0$ *such that*

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\frac{\mu(p)}{2}}{(\log\ )^{\sigma(p,d)}}\right)$$

(iv) (… 23) *Statement (iii) still holds for* $p = p_c$.

… 1 …

2.3. The Shnirelman theorem and the Quantum Unique ergodicity conjecture .

… $|\phi_k(x)|^2 d\,\mathrm{Vol}(x)$. … $k \longrightarrow +\infty$) … $|\phi_k(x)|^2 d\,\mathrm{Vol}(x)$

“ ” ( ). - fi fl , .

, L^1- $a : S^*M \longrightarrow \mathbb{R}$, - $(x_0; \ _0) \in S^*M$, $\frac{1}{T}\int_0^T a \circ g^t(x_0; \ _0)dt$ $T \longrightarrow +\infty$ - $\int_{S^*M} a\, dL$ L S^*M (. . ,) T^*M.

Quantum Ergodicity Theorem (Shnirelman theorem).

Theorem 5 (, , 10 , 40, 11)**.** *Let $(M; g)$ be a compact Riemannian manifold, with the metric normalized so that $\mathrm{Vol}(M) = 1$. Call Δ the Laplace-Beltrami operator on M. Assume that the geodesic flow of M is with respect to the Liouville measure. Let $(\phi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M; g)$ made of eigenfunctions of the Laplacian*

$$\Delta\phi_k = - \ _k\phi_k; \qquad _k \leq \ _{k+1} \longrightarrow +\infty:$$

Let a be a continuous function on M. Then

$$\frac{1}{N(\ )}\sum_{k,\lambda_k\leq\lambda}\left|\langle\phi_k; a\phi_k\rangle_{L^2(M)} - \int_M a(x)d\,\mathrm{Vol}(x)\right|^2 \underset{\lambda\longrightarrow+\infty}{\longrightarrow} 0 \tag{3}$$

where the normalizing factor is $N(\ ) = |\{k; \ _k \leq \ \}|$:

$\langle\phi_k; a\phi_k\rangle_{L^2(M)} = \int_M a(x)|\phi_k(x)|^2 d\,\mathrm{Vol}(x)$:

Remark 6. *The Cesaro limit* (3) *implies that there exists a subset $S \subset \mathbb{N}$ of density* 1 *such that*

$$\langle\phi_k; a\phi_k\rangle \underset{n\longrightarrow+\infty, n\in S}{\longrightarrow} \int_M a(x)d\,\mathrm{Vol}(x): \tag{4}$$

In addition, using the fact that the space of continuous functions is separable, one can actually find $S \subset \mathbb{N}$ of density 1 *such that* (4) *holds for $a \in C^0(M)$. In other words, the sequence of measures $(|\phi_k(x)|^2 d\,\mathrm{Vol}(x))_{n\in S}$ converges weakly to the uniform measure $d\,\mathrm{Vol}(x)$.*

Actually, the full statement of the theorem says that there exists a subset $S \subset \mathbb{N}$ of density 1 *such that*

$$\langle\phi_k; A\phi_k\rangle \underset{n\longrightarrow+\infty, n\in S}{\longrightarrow} \int_{S^*M} \ ^0(A)dL \tag{ }$$

for every pseudodifferential operator A of order 0 *on M. On the right-hand side, $^0(A)$ is the principal symbol of A, that is a function on the unit cotangent bundle S^*M. Equation 4 corresponds to the case where A is the operator of multiplication by the function a.*

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$k \longrightarrow +\infty$, , 0
1, 3 .

Quantum Unique Ergodicity conjecture. -
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M *negative sectional curvatures,* ( ),
$\longrightarrow +\infty$.
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fi

2. Entropy and support of semiclassical measures. , M
d.
$\langle \phi_n; A\phi_n \rangle$ ( ), A
ff 0. ,
$\langle \phi_{n_k}; A\phi_{n_k} \rangle$ A.
$\int_{S^*M} {}^0(A)d$, S^*M.
, " ", " ",

ϕ_{n_k}.

(ϕ_{n_k}).

$g^t_\sharp$ = $t \in \mathbb{R}$.

$\longrightarrow +\infty$

2.1.

Theorem 7. 3 *Assume M is a compact Riemannian manifold with negative sectional curvature. Assume $\langle \phi_{n_k}, A\phi_{n_k}\rangle$ converges to $\int_{S^*M}$ $^0(A)d$ for all A. Then has positive entropy.*

$h_{KS}(\ )$,

- $h_{KS}(\ ) = 0$
- $\mapsto h_{KS}(\ )$ $h_{KS}(\ _1 + (1-\ )\ _2) =\ h_{KS}(\ _1) + (1-\ )h_{KS}(\ _2)$, $_1;\ _2$, $\in [0;1]$
- (

$$( )\qquad h_{KS}(\ ) \leq \int \left(\sum_{j=1}^{r} \ ^+_j(\ ) \right) d\ (\ )$$

$^+_j(\ )$

- ( ) L (1, 2)
- d (S^*M $2d-1$), $d-1$ 1. ( )

$$( )\qquad h_{KS}(\ ) \leq d - 1;$$

L.

Corollary 1. *Let $\Gamma \subset S^*M$ be the union of all points lying on a periodic trajectory on the geodesic flow (recall, if M has negative curvature, there are countably many periodic geodesics). Let be as in Theorem 7. Then $(\Gamma) < 1$.*

scar (,).

strongly scarred

(, partial scar, $(\Gamma) > 0$).

Corollary 2. *The support of has Hausdorff dimension* > 1.

≥ 1

Theorem 8 (-). *Assume M is a compact Riemannian manifold of dimension d, with constant sectional curvature -1. Then has entropy greater than $\frac{d-1}{2}$.*

, $d-1$,

$1/2$. $\frac{d-1}{2}$

Corollary 3. *The support of has Hausdorff dimension* $\geq d$.

, T^*M $2d$, S^*M $2d-1$.

Corollary 4. *Let $\Gamma \subset S^*M$ be the union of all points lying on a closed trajectory on the geodesic flow. Let be as in Theorem 7, with M of constant negative curvature. Then $(\Gamma) \leq 1/2$.*

$= {}_1 + (1 -)\,{}_2$, $_1$ Γ, $h_{KS}(\,_1) = 0$ $h_{KS}(\,) = (1-\,)h_{KS}(\,_2)$. $\geq \frac{d-1}{2}$, $_2$ $d-1$, $\leq 1/2$.

$\frac{1}{2}\int_{S^*M}\sum_{j=1}^{d-1} {}^{+}_{j}\, d$, $^{+}_{j}$

$d = 2$, $1/2$

Theorem 9 (-). *has full support, that is, $(\Omega) > 0$ for any non-empty open set $\Omega \subset S^*M$.*

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2. . Some questions on non-compact manifolds.

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2. .1. *Absolutely continuous spectrum* . fi , " -
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L^2- , . ,
X *variable* ,
$M = \widetilde{X}$,
M
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2. .2. *Large frequency delocalization on non-compact manifolds.* ϕ_λ
$-\Delta\phi_\lambda = \phi_\lambda$ M ($\in \mathbb{R}$)
$|\phi_\lambda(x)|^2 d\,\mathrm{Vol}(x)$, $\longrightarrow +\infty$, M - . -

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ff

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L^2-

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$|\phi_\lambda(x)|^2 d\,\mathrm{Vol}(x)$

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$|\phi_\lambda(x)|^2 d\,\mathrm{Vol}(x)$.

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$C_c^\infty(M)$

102, 32 . , . ,

-L^2- " ".

ffi .

[redacted] CALE DELOCALI A ION

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spectral statistics -

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integrable dynamics *chaotic dynamics* 21, 2 , fi -

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$\Delta = \frac{d^2}{dx^2}$

" ff"

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fi ,

$\longrightarrow +\infty$ 41 . 1 , 4, , 1

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fi (" ").

regular ,

3.4 .

3.1. Overview of the problem. $G = (V, E)$.

localized, *delocalized*

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o fi (. -

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, . $I \subset \mathbb{R}$,

- *spectral localization :* I,
- *exponential localization :* ,
- *dynamical localization :* I

  .

  , ff ,
- *spectral delocalization :* I,
- *ballistic transport :* I fi

  (,) .

spatial delocalization. -

-

" " fi

(G_N) fi

(G_N) $N \to \infty$. , ,

(G_N) , fi

. fi

fi .

,

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. M_N $N \times N$ $(\phi_j)_{j=1}^N$

. ϕ_j fi

$\sum_{x=1}^N |\phi_j(x)|^2$,

$1/N$.

- ℓ^∞ *norms :* $|\phi_j(x)|$, ,

  $\|\phi_j\|_\infty$ $1/\sqrt{N}$
- ℓ^p *norms:* $\|\phi_j\|_p$ $N^{1/p-1/2}$ 1 , ϕ_j

  non-ergodic (*multi-fractal*) $\|\phi_j\|_p$ $N^{f(p)}$ $f(p) \neq 1/p - 1/2$.
- *Scarring :* $(\sum_{x\in\Lambda} |\phi_j(x)|^2 \geq 1 - \epsilon)$

  $(\sum_{x\in\Lambda} |\phi_j(x)|^2 \geq \epsilon)$ Λ " "

  " " .
- *Quantum ergodicity :* $a : \{1, \ldots, N\} \longrightarrow \mathbb{C}$,

  $\sum_x a(x)|\phi_j(x)|^2$ $\frac{1}{N}\sum_x a(x)$

  , .

most j. *all* j, *quantum unique ergodicity.*

almost sure ℓ^∞-

random *random*

2, 3, 33, 1, 1 , 13, 14 4.1.

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3.2. **Entropy.** 3 $(q+1)$-

(, ,

$q+1$).

$(q+1)$- $(G_N)_{N\in\mathbb{N}} = (V_N; E_N)$.

V_N

$$\mathcal{A}_N f(x) = \sum_{x\sim y} f(y) \quad ( )$$

$x \sim y$ x y .

$$\Delta_N f(x) = \sum_{x\sim y} (f(y) - f(x)): \quad ( )$$

$$\mathcal{A}_N - (q+1)I = \Delta_N: \quad (10)$$

Theorem 10 (- 3)**.** *Let* (G_N) *be a sequence of* $(q+1)$*-regular graphs (with* q *fixed),* $G_N = (V_N; E_N)$ *with* $V_N = \{1; : : : ; N\}$*. Assume that*[1] *there exists* $c > 0;$ > 0 *such that, for any* $k \le c \ln N$*, for any pair of vertices* $x; y \in V_N$*,*

$$|\{\text{paths of length } k \text{ in } G_N \text{ from } x \text{ to } y\}| \le q^{k\left(\frac{1-\delta}{2}\right)}: \quad (11)$$

Fix $\epsilon > 0$*. Then, if* ϕ *is an eigenfunction of the discrete Laplacian on* G_N *and if* $\Lambda \subset V_N$ *is a set such that*

$$\sum_{x\in\Lambda} |\phi(x)|^2 \ge \epsilon \sum_{x\in V_N} |\phi(x)|^2;$$

then $|\Lambda| \ge N^\alpha$ *— where* > 0 *is given as an explicit function of* $\epsilon;$ *and* c*.*

$$H_N(\phi) = -\frac{1}{\log N} \sum_x |\phi(x)|^2 \ln |\phi(x)|^2$$

[1] $\ge c \ln N$

.

L^∞- $\|\phi\|_\infty = O((\log N)^{-1/4})$. , 1=4 (11) 3 .

3.3. QE on regular graphs. fi , $G_N = (V_N; E_N)$ $(q+1)$- ,

(EXP) . , > 0 $(q+1)^{-1}\mathcal{A}_N$ $\ell^2(V_N)$ $\{1\} \cup [-1+ \; ; 1- \;]$ N.

1 , . - G_N - . - $\mathcal{A}_N$ 0, (3).

(BST) R,

$$\frac{|\{x \in V_N;\ (x) < R\}|}{N} \underset{N \longrightarrow \infty}{\longrightarrow} 0$$

(x) x, , r $B(x;r)$.

(BST) , - 1 , $(q+1)$- . , fi fi . $\mathfrak{X}$ $(q+1)$- . **(BST)** - . ($^{(N)}_1; \dots; ^{(N)}_N$) $\mathcal{A}_N$ G_N , $I \subset \mathbb{R}$,

$$\frac{1}{N}|\{j;\ ^{(N)}_j \in I\}| \underset{N \longrightarrow +\infty}{\longrightarrow} \int_I \mathrm{m}(\ )d$$

$\mathrm{m}(\ )$ $_o$ $\mathcal{A}$ $\ell^2(\mathfrak{X})$. Z ,

(12) $$\int {}^k\mathrm{m}(\ )d \ = \langle \ _o; \mathcal{A}^k_{\mathfrak{X}}\ _o\rangle_{\ell^2(\mathfrak{X})}$$

$\mathcal{A}_{\mathfrak{X}}$ $\mathfrak{X}$ $\mathfrak{X}$, o o k . , m , $(-2\sqrt{q}; 2\sqrt{q})$. $^{(N)}_j$ $(-2\sqrt{q}; 2\sqrt{q})$, $[-(q+1); q+1]$.

, 11.

Theorem 11 ( ). *Let $(G_N) = (V_N; E_N)$ be a sequence of $(q+1)$-regular graphs with $|V_N| = N$. Assume that (G_N) satisfies* **(BST)** *and* **(EXP)**.

Let $(\phi_1^{(N)}; \dots; \phi_N^{(N)})$ be an orthonormal basis of eigenfunctions of $\mathcal{A}_N$ in $\ell^2(V_N)$.

Let $a_N : V_N \longrightarrow \mathbb{C}$ be a sequence of functions such that $\sup_N \sup_{x\in V_N} |a_N(x)| \leq 1$. Define $\langle a_N \rangle = \frac{1}{N}\sum_{x\in V_N} a_N(x)$.

Then

$$\frac{1}{N}\sum_{j=1}^{N}\left|\langle \phi_j^{(N)}; a_N \phi_j^{(N)}\rangle_{\ell^2(V_N)} - \langle a_N\rangle\right|^2 \underset{N\longrightarrow+\infty}{\longrightarrow} 0.$$

, > 0,

$$\frac{1}{N}\left|\left\{ j \in [1; N]; \left|\langle \phi_j^{(N)}; a_N \phi_j^{(N)}\rangle_{\ell^2(V_N)} - \langle a_N\rangle\right| > \right\}\right| \underset{N\longrightarrow+\infty}{\longrightarrow} 0. \quad (13)$$

$\langle \phi_j^{(N)}; a_N \phi_j^{(N)}\rangle_{\ell^2(V_N)}$ $\phi_j^{(N)}$ $a_N \phi_j^{(N)}$, $\sum_{x\in V_N} a_N(x)|\phi_j^{(N)}(x)|^2$. 11 V_N,

$$\sum_{x\in V_N} |\phi_j^{(N)}(x)|^2 \; x \qquad \frac{1}{N}\sum_{x\in V_N} x \qquad (\quad)$$

(a_N). (13) N j, .

3.4. Non-regular graphs : from spectral to spatial delocalization.

regular . , , fi . *quantum ergodicity theorem* o . *spectral delocalization* fi *spatial delocalization* fi . ( ) ,

"If a large finite system is close (in the Benjamini-Schramm topology) to an infinite system having purely absolutely continuous spectrum in an interval I, then the eigenfunctions (with eigenvalues lying in I) of the finite system satisfy quantum ergodicity."

$(G_N)_{N\in\mathbb{N}}$. 3 d . V_N E_N G_N, . $|V_N| = N$ $N \longrightarrow \infty$. fi $\mathcal{A}_N : \mathbb{C}^{V_N} \to \mathbb{C}^{V_N}$

$$(\mathcal{A}_N f)(v) = \sum_{w\sim v} f(w);$$

$v \sim w$ v w . $\mathcal{A}_N$ (/) $N \longrightarrow +\infty$. ,

(EXP) (G_N) .
, - , fi ( ) $P_N : \mathbb{C}^{V_N} \to \mathbb{C}^{V_N}$

$$(P_N f)(x) = \frac{1}{d_N(x)} \sum_{y \sim x} f(y)\,; \tag{14}$$

$d_N(x)$ x. **(EXP)**
P_N , , 1 P_N
P_N $[-1+\ ;1-\ ] \cup \{1\}$, >0 N.

1 , . -
G_N - . -
P_N
0 (44, 3).
(G_N)

(BST) $r > 0$,

$$\lim_{N\to\infty} \frac{|\{x \in V_N :\ _{G_N}(x) < r\}|}{N} = 0\,;$$

$_{G_N}(x)$ *injectivity radius* x, . $B_{G_N}(x;\ )$
.

- (1),
(G_N),
rooted graphs ( ). ,
(BST) .

(BSCT) (G_N) *tree* $[\mathcal{T};o]$.
$\mathbb{P}$ $\{[\mathcal{T};o]\}$ $\mathbb{P}$

$(\ _j^{(N)})_{j=1}^N$ $\mathcal{A}_N$ $\ell^2(V_N)$. **(BSCT)** -
$: \mathbb{R} \longrightarrow \mathbb{R}$,

$$\frac{1}{N}\sum_{j=1}^{N} (\ _j^{(N)}) \underset{N\longrightarrow+\infty}{\longrightarrow} \mathbb{E}\left(\langle\ _o;\ (\mathcal{A}_\mathcal{T})\ _o\rangle\right) =: \int (t)\,d\mathrm{m}(t)\,; \tag{1\ }$$

$\mathcal{A}_\mathcal{T}$ $\mathcal{T}$, $\ell^2(\mathcal{T})$. $\mathbb{E}$
$\mathbb{P}$. m *integrated density of states*
o .

- -
- I $\mathcal{A}_\mathcal{T}$
($\mathbb{P}$ $[\mathcal{T};o]$). $[\mathcal{T};o]$ (
$\mathbb{P}$). $x;y \in \mathcal{T}$, $\in \mathbb{C}\setminus\mathbb{R}$,

$$\mathcal{G}^\gamma(x;y) = \langle\ _x;(\mathcal{A}_\mathcal{T} -\ )^{-1}\ _y\rangle_{\ell^2(\mathcal{T})}\,.$$

$v, w \in \mathcal{T}$ $v \sim w$, $\mathcal{T}^{(v|w)}$
$\mathcal{T}$ v w. $\mathcal{A}_{\mathcal{T}(\ |\ )}$
$\mathcal{G}^{(v|w)}(\cdot;\cdot;\ )$.
$_w^\gamma(v) := -\mathcal{G}^{(v|w)}(v; v;\ )$.

(Green) - I, $s > 0$

$$\sup_{\lambda \in I, \eta_0 \in (0,1)} \mathbb{E}\left(\sum_{y: y\sim o} |\operatorname{Im}\ _o^{\lambda+i\eta_0}(y)|^{-s}\right) < \infty:$$

(Green), fi ( )
$[\mathcal{T}; o]$

(1) $_o(J) = \langle\ _o; \mathbb{1}_J(\mathcal{A}_\mathcal{T})\ _o\rangle$ $J \subseteq \mathbb{R}:$

(Green) $\sup_{\lambda\in I, \eta_0>0} \mathbb{E}(|\mathcal{G}^\gamma(o;o)|^2) < \infty$.
, $\mathbb{P}$. . $[\mathcal{T}; o]$, $_o$ -
I, $\frac{1}{\pi}\operatorname{Im}\mathcal{G}^{\lambda+i0}(o;o)$. , **(Green)** $\mathbb{P}$. .
$\mathcal{A}_\mathcal{T}$ I. -
12 .
(Green), $s < 0$,
2 , ,
- o .
.
I **(Green)**, fi I_1 (fi
) $\bar{I}_1 \subset I$. G_N $\Gamma_N \backslash \widetilde{G_N}$ $\widetilde{G_N}$
($\widetilde{G_N}$). $\tilde{x}, \tilde{y}$ $\widetilde{G_N}$ $\in \mathbb{C}\setminus\mathbb{R}$,
$\widetilde{\mathcal{A}}_N$ $\widetilde{G_N}$

(1) $$\tilde{g}_N^\gamma(\tilde{x};\tilde{y}) = \langle\ _{\tilde{x}}; (\widetilde{\mathcal{A}}_N -\ )^{-1}\ _{\tilde{y}}\rangle_{\ell^2(\widetilde{G_N})}:$$

Theorem 12 (-). *Assume that* $(G_N; W_N)$ *satisfies* ***(BSCT)***, ***(EXP)*** *and* ***(Green)***.

Call $(\ _j^{(N)})_{j=1}^N$ *the eigenvalues of* $\mathcal{A}_N$ *on* $\ell^2(V_N)$, *and let* $(\phi_j^{(N)})_{j=1}^N$ *be a corresponding orthonormal eigenbasis.*

For each N, *let* $a = a_N$ *be a function on* V_N *with* $\sup_N \sup_{x\in V_N} |a_N(x)| \le 1$.

Then

$$\lim_{\eta_0\downarrow 0}\lim_{N\to+\infty}\frac{1}{N}\sum_{\lambda_j^{(N)}\in I_1}\left|\sum_{x\in V_N} a(x)|\phi_j^{(N)}(x)|^2 - \sum_{x\in V_N} a(x)\ ^N_{\lambda_j^{(N)}+i\eta_0}(x)\right| = 0;$$

for some family of probability measures $^N_\gamma$ *on* V_N, *indexed by a parameter* $\in \mathbb{C}\setminus\mathbb{R}$, *defined as follows :*

$$^N_\gamma(x) = \frac{\operatorname{Im}\tilde{g}_N^\gamma(\tilde{x};\tilde{x})}{\sum_{y\in V_N}\operatorname{Im}\tilde{g}_N^\gamma(\tilde{y};\tilde{y})}:$$

Here, $\tilde{x} \in \widetilde{G_N}$ *is a lift of* $x \in V_N$.

, $\epsilon > 0$,

$$(1\ )\quad \frac{1}{N}\left|\left\{ {}^{(N)}_{j} \in I_1 : \left|\sum_{x\in V_N} a(x)|\phi_j^{(N)}(x)|^2 - \sum_{x\in V_N} a(x)\ {}^{N}_{\lambda_j^{(N)}+i\eta_0}(x)\right| > \epsilon\right\}\right| \underset{N\to+\infty,\, \eta_0\downarrow 0}{\longrightarrow} 0:$$

12 ${}^{N}_{\gamma}$

. $a_N = 1\!\!1_{\Lambda_N}$,

$\Lambda_N \subset V_N$ $\approx$ N $\in (0;1)$. (G_N) ,

$\widetilde{G_N}$ N ($(q+1)$-)

$\tilde{g}_N^{\gamma}(\tilde{x};\tilde{y})$ $\mathcal{G}^{\gamma}(\tilde{x};\tilde{y})$. ,

$\mathcal{G}^{\gamma}(\tilde{x};\tilde{x}) = \mathcal{G}^{\gamma}(o;o)$ $\tilde{x} \in \widetilde{G_N}$. ${}^{N}_{\gamma}$

V_N ( ). (1) $\|1\!\!1_{\Lambda_N}\phi_j^{(N)}\|^2 \approx$ $\phi_j^{(N)}$.

$\phi_j^{(N)}$, $\Lambda_N \subset V_N$

, fi $|\phi_j^{(N)}|^2$.

, ${}^{N}_{\gamma}(\Lambda_N) =$ $|\Lambda_N| \approx$ N. ,

$c_\alpha > 0$ $\Lambda_N \subset V_N$ $|\Lambda_N| \geq$ N,

$$(1\ )\quad \inf_{\eta_0\in(0,1)} \liminf_{N\longrightarrow\infty} \inf_{\lambda\in I} {}^{N}_{\lambda+i\eta_0}(\Lambda_N) \geq 2c_\alpha:$$

(1),

Corollary 13. *For any* $\in (0;1)$, *there exists* $c_\alpha > 0$ *such that for any* $\Lambda_N \subset V_N$ *with* $|\Lambda_N| \geq$ N, *we have*

$$\frac{1}{N}\#\left\{ {}^{(N)}_{j} \in I : \left\|1\!\!1_{\Lambda_N}\phi_j^{(N)}\right\|^2 < c_\alpha\right\} \underset{N\longrightarrow+\infty}{\longrightarrow} 0:$$

, $\|1\!\!1_{\Lambda_N}\phi_j^{(N)}\|^2 \approx$ $\phi_j^{(N)}$,

, $\|1\!\!1_{\Lambda_N}\phi_j^{(N)}\|^2 \geq c_\alpha > 0$ $\phi_j^{(N)}$.

.

$F:\mathbb{R}\to\mathbb{R}$, $\in I$,

$$(20)\quad \frac{1}{N}\sum_{x\in V_N} F\left(N\ {}^{N}_{\lambda+i\eta_0}(x)\right) \underset{N\longrightarrow+\infty}{\longrightarrow} \mathbb{E}\left(F\left(\frac{\operatorname{Im}\mathcal{G}^{\lambda+i\eta_0}(o;o)}{\mathbb{E}\left(\operatorname{Im}\mathcal{G}^{\lambda+i\eta_0}(o;o)\right)}\right)\right):$$

$\left(N\ {}^{N}_{\lambda+i\eta_0}(x)\right)$ (x

V_N) $\left(\frac{\operatorname{Im}\mathcal{G}^{\gamma}(o,o)}{\mathbb{E}(\operatorname{Im}\mathcal{G}^{\gamma}(o,o))}\right)$.

${}^{N}_{\lambda+i\eta_0}(x)$ $1{=}N$ N, - .

Remark 14. *The results proven in actually hold for more general Schrödinger operators than adjacency matrices : one can consider weighted Laplacians (with conductances on the edges) and add a potential; in other words, on each* G_N, *we can consider a discrete Schrödinger operator* $\mathcal{H}_N$. *The limiting object in assumption* ***(BSCT)*** *is now a random* *rooted tree $[\mathcal{T}; o]$ endowed with a random Schrödinger operator $\mathcal{H}$. Assumption* ***(Green)*** *has to be modified, replacing the adjacency matrix $\mathcal{A}$ by the operator $\mathcal{H}$. Similarly, in the statement of the theorem, the Green functions $\tilde{g}_N^\gamma$ to be considered are those of $\mathcal{H}_N$ lifted to the universal cover $\widetilde{G_N}$.*

Remark 15. *In particular, our result applies to the case where the limiting system $([\mathcal{T}; o]; \mathcal{H})$ is $\mathcal{T} = \mathfrak{X}$ (the $(q+1)$-regular tree) with an arbitrary origin o, and $\mathcal{H} = \mathcal{H}_\epsilon = \mathcal{A} + \epsilon\mathcal{W}$ where $\mathcal{W}$ is a random real-valued potential on $\mathfrak{X}$. More precisely the values $\mathcal{W}(x)$ $(x \in \mathfrak{X})$ are i.i.d. random variables of common law . This is known as the Anderson model on $\mathfrak{X}$. It was shown by A. Klein that the spectrum of $\mathcal{H}_\epsilon$ is a.s. purely absolutely continuous on $I = (-2\sqrt{q} + \ ; 2\sqrt{q} - \ )$, provided ϵ is small enough (depending on). This just assumes a second moment on . Under stronger regularity assumptions on , one can show that Assumption* ***(Green)*** *holds on I (see 11 , following Aizenman-Warzel 2). Examples of sequences of expander regular graphs G_N with discrete Schrödinger operators $\mathcal{H}_N$ converging to $([\mathfrak{X}; o]; \mathcal{H}_\epsilon)$ are given in 10 .*

Remark 16. *Examples of sequences of - satisfying our three assumptions were investigated in 11 . In the examples considered there, the limiting trees $\mathcal{T}$ are fi ; roughly speaking, those are trees where the local geometry can only take a finite number of values. If $\mathcal{A}$ is the adjacency matrix of such a tree, we showed in 11 that the spectrum of $\mathcal{A}$ is a finite union of closed intervals, and that there are a finite number of points $y_1; \dots; y_\ell$ in such that Assumption* ***(Green)*** *holds on any I of the form $\setminus ([y_1 - \ ; y_1 + \ ] \cup \dots \cup [y_\ell - \ ; y_\ell + \ ])$ (for any > 0). We showed – extending Remark 15 – that on such trees, Assumption* ***(Green)*** *remains true after adding a small random potential to $\mathcal{A}_\mathcal{T}$. Finally, we showed the existence of sequences (G_N) converging to $\mathcal{T}$ and satisfying the* ***(EXP)*** *condition.*

[redacted] AND LINK I H O HE O K

4.1. Random regular graphs. 11 *deterministic* . (G_N) , a. , " " , " " a. , - *random* o . *random* , fi 113, 4 , 1 , 13 fi , 14 , . . . ( ) - N $\{1; \dots; N\}$, , $(q+1)$- ( ) $(q+1)N$. fi . 1 , 13, 14 " " " *adjacency matrix* $a_N : \{1; \dots; N\} \longrightarrow \mathbb{R}$,

$(q+1)$- $\{1;:::;N\}$ $\sum_{x=1}^{N} a_N(x)|\phi_j^{(N)}(x)|^2$
$\langle a_N \rangle$ *all* j,

Theorem 17 (- 14). *Let ! be such that* $\sqrt{q} \geq (! + 1)2^{2\omega+45}$.

(i) With probability $\geq 1 - o(N^{-\omega+8})$ *on the choice of the graph,*

$$\|\phi_j\|_\infty \leq \frac{(\log N)^{121}}{\sqrt{N}}$$

for all eigenfunctions associated to eigenvalues such that $|\ _j \pm 2\sqrt{q}| > (\log N)^{-3/2}$.

(ii) (Quantum Unique Ergodicity for random regular graphs) Given an observable a_N : $\{1;:::;N\} \longrightarrow \mathbb{R}$, *we have, with probability* $\geq 1 - o(N^{-\omega+8})$ *on the choice of the graph, for* N *large enough,*

$$\left|\sum_{x=1}^{N} a_N(x)|\phi_j^{(N)}(x)|^2 - \langle a_N \rangle\right| \leq \frac{(\log N)^{250}}{N}\sqrt{\sum_x |a_N(x)|^2}; \tag{21}$$

for all eigenfunctions associated to eigenvalues $\ _j \in (-2\sqrt{q}+\epsilon; 2\sqrt{q}-\epsilon)$ *(bulk eigenvalues).*

In particular, if $a_N = 1\!\!1_{\Lambda_N}$ *where* $\Lambda_N \subset \{1;:::;N\}$, *we find*

$$\left|\sum_{x\in\Lambda_N} |\phi_j^{(N)}(x)|^2 - \frac{|\Lambda_N|}{N}\right| \leq \frac{(\log N)^{250}}{N}\sqrt{|\Lambda_N|}:$$

($! > 8$ $q > 2^{128}$).

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$|\Lambda_N| > (\log N)^{500}$. , 11,

∞,

$$\left|\sum_{x\in\Lambda_N} |\phi_j^{(N)}(x)|^2 - \frac{|\Lambda_N|}{N}\right| \leq \frac{1}{\sqrt{N \log N}}\sqrt{|\Lambda_N|}:$$

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most j $|\Lambda_N| > N^{1/2}$. , ,

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1 *almost sure* .

Remark 18. *Note that we emphasized Theorem 17 from* 14 *because our main concern here is the delocalization of eigenfunctions. The main focus of* 14 *is however on the universality of the local spectral statistics for random regular graphs. This would deserve a separate paper.*

12 ,

G_N N *all* $\phi_j^{(N)}$,

$\sqrt{N}\phi_j(x)$ x $\{1;:::;N\}$ $\mathcal{N}(0;\ {}_j^2)$ $0 \leq\ {}_j \leq 1.$ $R \geq 0,$ x $\{1;:::;N\}$ $(\sqrt{N}\phi_j(y))_{y,d(y,x)\leq R}$ x $G_N,$ $B_{\mathfrak{X}}(o;R).$ ${}_j^2\Phi_{\lambda_j}(d(x;y))$ Φ_{λ_j} ${}_j$ $(q+1)$- $\mathfrak{X}.$ ${}_j - 1,$ ${}_j \neq 0,$ $o(N)$. 10 , N^{α} . 11, 1 , a_N . j (1) , (21), a_N. ($N)$, (1) (21).

4.2. From graph Laplacians to Hecke operators.

11 fi fi , . , , 11. ff , ". $\mathbb{S}^2$, 2- , . $\Delta_{\mathbb{S}^2}$, . . $\mathbb{S}^2 \subset \mathbb{R}^3$ 3 . $\Delta_{\mathbb{S}^2}$ $-\ell(\ell+1)$ ($2\ell+1$). $\Delta_{\mathbb{S}^2}$ fi $\mathbf{J}_{12} = \frac{1}{i}\left(x_2\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right).$ J_{12} ff 1, .

$(\phi_n) = (Y_\ell^m)_{\ell\geq 0,|m|\leq\ell}$ $\Delta_{\mathbb{S}^2}$ J_{12} . , 2.4, (ϕ_{n_k}) $\langle\phi_{n_k};A\phi_{n_k}\rangle$ A $\int_{S^*M}\ {}^0(A)d$, S^*M. ϕ_{n_k} - J_{12} (S^*M) .

, . 11 . " fi (114).

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$\Delta_{\mathbb{S}^2}$,

$\mathbb{S}^2$ $g_1, \ldots, g_k$ $SO(3)$,

$$T_k f(x) = \sum_{j=1}^{k} \left(f(g_j x) + f(g_j^{-1} x) \right)$$

$\Delta_{\mathbb{S}^2}$.

Theorem 19 (, 3). *Assume that $g_1, \ldots, g_k$ generate a free subgroup of $SO(3)$.*

For each ℓ, let $(\psi_j^{(\ell)})_{j=1}^{2\ell+1}$ be an orthonormal family of eigenfunctions of $-\Delta_{\mathbb{S}^2}$ of eigenvalue $\ell(\ell+1)$, that are also eigenfunctions of T_k.

Then for any continuous function a on $\mathbb{S}^2$, we have

$$\frac{1}{2\ell+1} \sum_{j=1}^{2\ell+1} \left| \int_M a(x) |\psi_j^{(\ell)}(x)|^2 d\mathrm{Vol}(x) - \int_M a(x) d\mathrm{Vol}(x) \right|^2 \underset{\ell \longrightarrow \infty}{\longrightarrow} 0.$$

T_k ℓ

T_k ℓ^{-1}.

$2k$- $N \sim \ell^2$ $g_1, \ldots, g_k$

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Remark 20. *T_k is not a pseudodifferential operator, so the argument sketched above to show that the basis $(Y_\ell^m)_{\ell \geq 0, |m| \leq \ell}$ could not satisfy quantum ergodicity does not apply here.*

Remark 21. *We note that for very special choices of rotations – rotations that correspond to norm n elements in an order in a quaternion division algebra, the operators T_k are called Hecke operators. It has been conjectured by Böcherer, Sarnak, and Schulze-Pillot 2 that such joint eigenfunctions satisfy the much stronger quantum unique ergodicity property. This conjecture is still open.*

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4.3. **Quantum ergodicity on Riemann surfaces of high genus.** 11 12

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Theorem 22 (– 0). *Let (S_N) be a sequence of hyperbolic surfaces, whose genus (equivalently, volume) goes to ∞.*

(EXP) *Assume the first eigenvalue $\lambda_1(N)$ of $-\Delta$ on S_N is bounded away from 0 as $N \longrightarrow \infty$.*

(BSH) *Assume there are few short geodesics; in other words, (S_N) converges in the Benjamini-Schramm sense to the hyperbolic disc : for any $R > 0$,*

$$\lim_{N \longrightarrow +\infty} \frac{\mathrm{Vol}\{x \in S_N : (x) < R\}}{\mathrm{Vol}(S_N)} = 0$$

where (x) means the injectivity radius at x.

Fix an interval $I \subset (1/4; +\infty)$.

Let $(\phi_i^{(N)})$ be an orthonormal basis of eigenfunctions of the Laplacian on S_N.

Let $a = a_N : S_N \longrightarrow \mathbb{C}$ be such that $|a(x)| \leq 1$ for all $x \in S_N$. Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{Vol(S_N)} \sum_{\lambda_i(N) \in I} \left| \int_{S_N} a(x) |\phi_i^{(N)}(x)|^2 dx - \langle a \rangle \right|^2 = 0$$

where $\langle a \rangle = \frac{1}{Vol(S_N)} \int_{S_N} a(x) dx$.

$(1/4; +\infty)$ L^2-

S_N

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