# Absolute purity in motivic homotopy theory

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joint work with F. Deglise, J. Fasel and A. Khan

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# The absolute purity conjecture

Grothendieck's **absolute (cohomological) purity conjecture** (SGA5, Expose I 3.1.4) is the following statement: if $i : Z \to X$ is a closed immersion between noetherian regular schemes of pure codimension $c$, $n \in \mathcal{O}(X)$ and $\quad = \mathbb{Z}=n\mathbb{Z}$, then the etale cohomology sheaf supported in $Z$ with values in $\quad$ can be computed as

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This conjecture has been solved by Gabber.

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  Based on Thomason's method + rigidity for algebraic $K$-theory

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- Our work: study absolute purity in the framework of motivic homotopy theory.
- Main result: the absolute purity in motivic homotopy theory is satis ed with rational coe cients in mixed characteristic.

# Motivic homotopy theory

- The motivic homotopy theory or $\mathbb{A}^1$-homotopy theory is introduced by Morel and Voevodsky (1998) as a framework to study cohomology theories in algebraic geometry, by importing tools from algebraic topology
- Idea: use the a ne line $\mathbb{A}^1$ as a substitute of the unit interval to get an algebraic version of the homotopy theory
- Can be used to study cohomology theories such as algebraic $K$-theory, Chow groups (motivic cohomology) and many others
- Advantage: has many a lot of structures coming from both topological and algebraic geometrical sides

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- $\mathbb{A}^1$-enumerative geometry (Hoyois, Levine, Kass-Wickelgren)
- Non-commutative geometry and singularity categories (Tabuada, Blanc-Robalo-Toen-Vezzosi)

# Some topological background

- A **spectrum** $\mathbb{E}$ is a sequence $(E_n)_{n \in \mathbb{N}}$ of pointed spaces (e.g. CW-complexes or simplicial sets) together with continuous maps $\ _n : S^1 \wedge E_n \to E_{n+1}$ called **suspension maps**

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- From an $\infty$-categorical point of view, the category of spectra is the stabilization of the category of spaces, and is the universal stable (triangulated) category

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- Bigraded $\mathbb{A}^1$-**homotopy sheaves**: for $X \in \mathbf{H}\ (S)$, $\ ^{\mathbb{A}^1}_{a,b}(X)$ is the Nisnevich sheaf on $Sm_S$ associated to the presheaf

$$U \mapsto [U \wedge S^{a\ \ b} \wedge \mathbb{G}_m^b; X]_{\mathbf{H}_\bullet(S)}$$

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- In the classical notation, $S^1 = \mathbb{1}[1]$ and $\mathbb{G}_m = \mathbb{1}(1)[1]$
- **SH**$(S)$ is the universal stable $\infty$-category which satisfies Nisnevich descent and $\mathbb{A}^1$-invariance (Robalo, Drew-Gallauer)

# Motivic spectra

Every object in $\mathbf{SH}(S)$ represents a bigraded cohomology theory

$$\mathbb{E}^{p,q}(U) = [U; (S^1)^{\wedge(p \quad q)} \wedge (\mathbb{G}_m)^{\wedge q} \wedge \mathbb{E}]_{\mathbf{SH}(S)}$$

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- Milnor-Witt spectrum $\mathbf{H}_{MW}\mathbb{Z}$ represents Milnor-Witt motivic cohomology/higher Chow-Witt groups (Deglise-Fasel)

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- The 1-line is also computed (Rondigs-Spitzweck- stvaer):

$$0 \to K_2^M \ _n = 24 \to \ _{n+1,n}(\mathbb{1}_k) \to \ _{n+1,n} f_0(\mathbf{KQ})$$

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For any separated morphism of  nite type $f : X \to Y$, there is an additional pair of adjoint functors

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- Originates from Grothendieck's theory for $l$-adic sheaves (SGA4), and worked out in the motivic setting by Ayoub and Cisinski-Deglise
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- They satisfy formal properties axiomatizing important theorems such as duality, base change and localization.

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- In the presence of an *orientation*, we recover the usual relative purity

# Orientations

- An **absolute motivic spectrum** is the data of $\mathbb{E}_X \in \mathbf{SH}(X)$ for every scheme $X$, together with natural isomorphisms $f\ \mathbb{E}_X \simeq \mathbb{E}_Y$ for every morphism $f : Y \to X$

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- A theory of *fundamental classes* aims at establishing a cohomological intersection theory
- For oriented spectra, Deglise de ned fundamental classes using Chern classes

# Bivariant groups

- For $f : X \to S$ be a separated morphism of nite type, $v \in K_0(X)$ and $\mathbb{E} \in \mathbf{SH}(S)$, de ne the $\mathbb{E}$-**bivariant groups** (or **Borel-Moore $\mathbb{E}$-homology**) as

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$$\mathbb{E}_n(X{=}S; v) = [f_! Th(v)[n]; \mathbb{E}]_{\mathbf{SH}(S)}$$

- If $S$ is a  eld and $\mathbb{E} = \mathbf{H}\mathbb{Z}$, then $\mathbb{E}_i(X{=}S; v) = CH_r(X; i)$ are the higher Chow groups, where $r$ is the virtual rank of $v$

# Functoriality of bivariant groups

- Base change:

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow{p} & S \end{array}$$

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- Product: if $\mathbb{E}$ has a ring structure, $X \xrightarrow{f} Y \xrightarrow{g} S$

$$\mathbb{E}_m(X{=}Y; w) \otimes \mathbb{E}_n(Y{=}S; v) \to \mathbb{E}_{m+n}(X{=}S; w + f\ \ v)$$

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  - **purity transformation** $f^* \otimes \mathrm{Th}(\ _f) \to f^!$
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  all compatible with compositions
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- The construction uses the *deformation to the normal cone*

# Euler class and excess intersection formula

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  where $p$ and $q$ are lci, we have $\;_p = \;_q \cdot e(\;)$, where is the *excess bundle*
- **Motivic Gauss-Bonnet formula** (Levine, Deglise-J.-Khan) For $p : X \to S$ a smooth and proper morphism

$$(X=S) = p \; e(T_p)$$

  where $(X=S)$ is the *categorical Euler characteristic*

# The absolute purity property

- We say that an absolute spectrum $\mathbb{E}$ satis es **absolute purity** if for any closed immersion $i : Z \to X$ between regular schemes, the purity transformation $\mathbb{E}_Z \otimes \mathrm{Th}( _f) \to f^!\mathbb{E}_X$ is an isomorphism

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- From this property Cisinski-Deglise deduce that the rational motivic Eilenberg-Mac Lane spectrum $\mathbf{H}\mathbb{Q}$ also satis es absolute purity, mainly because $\mathbf{H}\mathbb{Q}$ is a direct summand of $\mathbf{KGL}_{\mathbb{Q}}$ by the Grothendieck-Riemann-Roch theorem

# The Main result

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- The \switching factors" endomorphism of $\mathbb{P}^1 \wedge \mathbb{P}^1$ induces a decomposition of the sphere spectrum $\mathbb{1}_{\mathbb{Q}}$ into the direct sum of the plus-part $\mathbb{1}_{+,\mathbb{Q}}$ and the minus-part $\mathbb{1}_{\ ,\mathbb{Q}}$ (Morel)

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- The $+$-part $\mathbb{1}_{+,\mathbb{Q}}$ agrees with $\mathbf{H}\mathbb{Q}$ (Cisinski-Deglise)
- Therefore it su ces to show that the minus part satis es aboslute purity

# The rst proof

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- This proves the absolute purity of $\mathbb{1}_{\mathbb{Q}}$ when 2 is invertible on the base scheme, since **KQ** is only well-de ned in this case

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- For $X$ a field of characteristic zero, $\ _X$ is automatically an isomorphism; for $X$ a field of positive characteristic, $\mathbf{SH}(X)_{\ ,\mathbb{Q}}$ vanishes by a theorem of Morel/Bachmann
- The key lemma then reduces the absolute purity of $\mathbb{1}_{\ ,\mathbb{Q}}$ in mixed characteristic to the case of $\mathbb{Q}$-schemes, which can be proved using Popescu's theorem: a closed immersion of affine regular schemes over a perfect field is a limit of closed immersions of smooth schemes

# Some applications

- Our method can be used to deduce the following new results in mixed characteristic:
  - The six functors preserve constructible objects in the rational stable motivic homotopy category $\mathbf{SH}(\cdot;\mathbb{Q})$
  - The Grothendieck-Verdier duality holds for $\mathbf{SH}(\cdot;\mathbb{Q})$
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- Related work: absolute purity of the sphere spectrum over a Dedekind domain (Frankland-Nguyen-Spitzweck, work in progress)

Thank you!